Financial price fluctuations in a stock market model with many interacting agents

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Abstract

We consider a financial market model with a large number of interacting agents. Investors are heterogeneous in their expectations about the future evolution of an asset price process. Their current expectation is based on the previous states of their "neighbors" and on a random signal about the "mood of the market." We analyze the asymptotics of both aggregate behavior and asset prices. We give sufficient conditions for the distribution of equilibrium prices to converge to a unique equilibrium, and provide a microeconomic foundation for the use of diffusion models in the analysis of financial price fluctuations.

Key Words: agent-based modelling, diffusion models for financial markets, contagion effects, bubbles and crashes.

JEL subject classification: D40, D84, G10

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1 Introduction

In mathematical finance, the price evolution of a risky asset is usually modelled as the trajectory of a diffusion process defined on some underlying probability space. Geometric Brownian motion is now widely used as the canonical reference model. As prices are generated by the demand and supply of market participants, this approach should be explained in terms of a microeconomic model of interacting agents. Bick (1987) showed that geometric Brownian motion can indeed be justified as the rational expectations equilibrium in a market with homogeneous agents who all believe in this kind of price dynamics, and who instantaneously discount all available information into the present price; see also Kreps (1982) and Borckett and Witt (1991). On the other hand, Brock and Hommes (1997), Gaunersdorfer (2000), Lux and Marchesi (2000) and Kirman (1998), among others, identified heterogeneity among traders as a key element affecting the dynamics of financial price fluctuations. Heterogeneity in financial markets arises naturally from different expectations about the future movement of asset prices or from access to diverse information sets. At the same time, market participants are not isolated units: their decisions are often importantly influenced by their observations of the behavior of other individuals or the prevailing mood of the market.

In recent years there has been an increasing interest in agent-based models for financial markets which account for imitation and contagion effects in the formation of asset prices. Day and Huang (1990), Lux (1995, 1998) and Brock and Hommes (1997, 1998) described price processes in the context of deterministic dynamical systems. These authors studied situations in which two types of traders interact in the market. The first type, *fundamentalists*, believes that the price of an asset is entirely determined by some underlying fundamental value. The second type, typically called *trend chasers* or *chartists*, tries to predict future asset prices through past observations. In their models endogenous switching between the different types of market participants can cause large and sudden price fluctuations. The fluctuations may even exhibit a chaotic behavior if the effects of trend chasing become too strong.

This paper provides a unified *probabilistic* framework within which to model stock price dynamics resulting from the interaction of a large number of traders. Following an approach suggested by Föllmer and Schweizer (1993) and Föllmer (1994), we view stock prices as a sequence of temporary price equilibria. The demand of the agent a in period t depends on his current individual state x_t^a reflecting, for example, his expectation about the stock price in the following period. The fluctuation in the distribution of individual states will be the only component affecting the formation of price equilibria. The microscopic process $\{x_t\}_{t\in\mathbb{N}}$ which describes the stochastic evolution of all the individual states is specified in terms of an interacting Markov chain. In models motivated by statistical physics one usually has in mind a local form of interaction; a mean-field interaction is typically viewed as a mere simplification to circumvent the deeper problems related to local interactions. But in an economic context, agents are often influenced by signals about aggregate quantities. This calls for an additional global component in the interaction. In the context of our financial market model, the local and global dependence in the individual transition laws captures the idea that agents' expectations about the future value of a risky asset may be influenced by both the previous expectations of some acquaintances and the prevailing mood of the market.

The mood of the market is described by the empirical distribution of individual agents' states or, more completely, by the empirical field $R(x)$ associated with the configuration x. The *microscopic process* $\{x_t\}_{t\in\mathbb{N}}$ generates, via the *macroscopic process* $\{R(x_t)\}_{t\in\mathbb{N}}$ a random medium $\{\tilde{\varrho}_t\}_{t\in\mathbb{N}}$ for the evolution of the asset price process. Specifically, the logarithmic price process $\{p_t\}_{t\in\mathbb{N}}$ obeys a linear recursive relation of the form

$$p_{t+1} = f(\tilde{\varrho})p_t + g(\tilde{\varrho}_t)$$

in a random environment of investor sentiment. If the mood of the market is already in equilibrium, then the long run behavior of stock prices can be studied using standard results from the theory of stochastic difference equations given in, e.g., Vervaat (1979), Brandt (1986) or Borovkov (1998). Economically, however, such a stationarity assumption on the random environment is very restrictive. It is more natural to investigate the dynamics of financial price fluctuations under the assumption that the mood of the market is out of equilibrium, but settles down in the long run.

In a first step we analyze the long run behavior of the macroscopic process. We show that the dynamics on the level of aggregate behavior can be described by a Markov chain. From this we deduce that the mood of the market settles down in the long run if the interaction between different agents is weak enough. In a second step we show that asymptotic stationarity of the mood of the market implies asymptotic stationarity of the induced asset price process if the effects of technical trading are on average not too strong. Finally, we derive a continuous-time approximation of our discrete-time price process. Proving a functional central limit theorem for stochastic processes evolving in a non-stationary random environment, we show that the discrete-time price process can be approximated in law by a diffusion model if the mood of the market is asymptotically stationary.

This paper summarizes the results in Horst (2000). We introduce our financial market model in Section 2 where we also study the dynamics of both individual and aggregate behavior. The asset price process is analyzed in Section 3.

2 The Microeconomic Model

We consider a financial market model with an infinite set $\mathbb{A}$ of interacting agents trading a single risky asset. Following Föllmer and Schweizer (1993), the price evolution of the asset will be described by a sequence $\{p_t\}_{t\in\mathbb{N}}$ of temporary price equilibria. In reaction to a proposed price p in period t the agent $a \in \mathbb{A}$ forms an *excess demand* $z_t^a(p)$. Individual excess demand is obtained by comparing the proposed price with some individual *reference level* p_t^a the agent adopts for period t. In this paper we study the simplest case where individual excess demand takes the log-linear form

$$z_t^a(p) := \log p_t^a - \log p. \tag{1}$$

All heterogeneity across agents is incorporated into reference levels. The quantity p_t^a depends on the *state* x_t^a of the agent a chosen from a finite set C. Specifically, individual benchmarks are given in terms of individual combinations of a *fundamentalist* and a *trend chasing* component as

$$\log p_t^a = \log p_{t-1} + \alpha(x_t^a)(\log F - \log p_{t-1}) + \beta(x_t^a)(\log p - \log p_{t-1}) \tag{2}$$

with non-negative coefficients $\alpha(x_t^a), \beta(x_t^a) \in [0, 1)$. This includes the case where the agents can choose between a pure fundamentalist and a pure trend chasing strategy. The expectation of a *fundamentalist*,

$$\log p_t^a = \log p_{t-1} + c_F(\log F - \log p_{t-1}) \qquad (c_F > 0), \tag{3}$$

is based on the idea that the next price will move closer to the fair value F of the stock. A *chartist*, on the other hand, takes the proposed price as a signal about the future evolution of stock prices:

$$\log p_t^a = \log p_{t-1} + c_C(\log p - \log p_{t-1}) \qquad (c_C > 0). \tag{4}$$

The quantities c_F and c_C may be viewed as a measure for the trading volume of an individual fundamentalist and chartist, respectively.

The specific structure of individual reference levels yields $z_t^a(p) = z(p, x_t^a)$ for some function $z : \mathbb{R} \times C \to \mathbb{R}$. The actual stock price will be determined by the *market clearing condition* of zero total excess demand. In equilibrium, i.e., for $p = p_t$, a chartist's forecast is based on a past price trend.

2.1 The dynamics of the price process

We focus on the effects the fluctuations in agents' characteristics have on the dynamics of asset prices. Thus, the *microscopic process* $\{x_t\}_{t\in\mathbb{N}}$, $x_t = (x_t^a)_{a\in\mathbb{A}}$, will be the only component affecting the formation of price equilibria. Its state space will be given by a suitable subset E_0 of the *configuration space* $E := C^{\mathbb{A}}$. For each $x \in E_0$, the weak limit

$$\varrho(x) := \lim_{n\to\infty} \frac{1}{|\mathbb{A}_n|} \sum_{a\in\mathbb{A}_n} \delta_{x^a}(\cdot) \in \mathcal{M}(C) \tag{5}$$

exists along a suitable sequence of finite sets $\mathbb{A}_n \uparrow \mathbb{A}$. Here $\mathcal{M}(C)$ denotes the class of all probability measures on C, and $\delta_{x^a}(\cdot)$ is the Dirac measure concentrated on x^a. In particular, the sequence of individual states $\{x_t\}_{t\in\mathbb{N}}$ will induce the sequence of *empirical distributions* $\{\varrho(x_t)\}_{t\in\mathbb{N}}$.

Definition 2.1 *We call $\varrho(x_t)$ the mood of the market in period t.*

In a financial market model with an infinite set of agents, it is reasonable to view the set of agents who are actually involved in the formation of successive prices as subsets of the much larger set of agents constituting the entire economy. Hence we assume that the empirical distribution $\tilde{\varrho}_t$ of the states assumed by those traders who are active on the market at time t is a random variable whose conditional law

$$\tilde{Q}(\varrho(x_t);\cdot) \tag{6}$$

is specified in terms of a stochastic kernel $\tilde{Q}$ on $\mathcal{M}(C)$[1]. The stock price p_t is then defined through the market clearing condition,

$$\int z(p_t, x)\tilde{\varrho}_t(dx) = 0, \tag{7}$$

of zero total excess demand. Introducing the aggregate quantities

$$f(\tilde{\varrho}) := \frac{1 + \int(\beta - \alpha)d\tilde{\varrho}}{\int \beta d\tilde{\varrho}} \qquad \text{and} \qquad g(\tilde{\varrho}) := \frac{\log F \int \alpha d\tilde{\varrho}}{\int \beta d\tilde{\varrho}},$$

we see that the dynamics of the logarithmic stock price process defined through (1), (2) and (7) is described by the linear recursive relation

$$\log p_{t+1} = f(\tilde{\varrho}_{t+1}) \log p_t + g(\tilde{\varrho}_{t+1}) \tag{8}$$

in a random environment $\{\tilde{\varrho}_t\}_{t\in\mathbb{N}}$ of investor sentiment. The environment descries the stochastic evolution of the mood of the market. Our goal is to derive conditions on the behavior of individual agents which guarantee that the price process has a unique limiting distribution.

Remark 2.2 *There is no reason to assume that the mood of the market is already in equilibrium, i.e., that the driving sequence $\{\tilde{\varrho}_t\}_{t\in\mathbb{N}}$ is ergodic. Hence we are naturally led to consider situations in which the price process evolves in a non-stationary random environment.*

At times where $|f(\tilde{\varrho}_t)| < 1$ stock prices behave in a recurrent manner. However, as illustrated by the following example, price fluctuations can become highly volatile in periods where the impact of technical trading becomes too strong. This feature can be viewed as the temporary occurrence of bubbles or crashes generated by trend chasing.

[1]Mathematically, this approach provides an additional smoothing effect. Under a mild technical condition on $\tilde{Q}$ that does not alter the quantitative behavior of asset prices, the sequence $\{\tilde{\varrho}_t\}_{t\in\mathbb{N}}$ has better asymptotic properties than the process $\{\varrho(x_t)\}_{t\in\mathbb{N}}$.

Example 2.3 *Assume that the agents can either follow fundamentalist or a trend chasing strategy, i.e., put* $C = \{0, 1\}$ *and consider the reference values defined in (3) and (4). Let* $\tilde{\varrho}^c_t$ *be the fraction of chartists in period* t*. Then*

$$f(\tilde{\varrho}) := \frac{1 - c_F(1 - 2\tilde{\varrho}^c)}{1 - c_C\tilde{\varrho}^c} \quad \textit{and} \quad g(\tilde{\varrho}) := \frac{c_F(1 - \tilde{\varrho}^c)\log F}{1 - c_C\tilde{\varrho}^c}.$$

If $c_C > 1$*, then the maps* f *and* g *have singularities. Asset prices become highly unstable if the actual fraction of chartist is close to the critical value* $\tilde{\varrho}^* = c_C^{-1}$*. More generally, the price process behaves in a transient manner in periods where the fraction of trend chasers is so large that*

$$\tilde{\varrho}^c_t > \frac{c_F}{2c_F + c_C}.$$

Hence both a small c_F *and a large* c_C *favors instability of stock prices. This result is in accordance with the findings in, e.g., Lux (1998).*

Despite the destabilizing effects the presence of chartists has on the formation of stock prices, we shall see that the overall behavior of the price process is ergodic if the impact of noise traders is on average not too strong, and if the interaction between different traders is weak enough.

2.2 The Dynamics of Individual Behavior

The microscopic process will be described by an interactive Markov chain,

$$\Pi(x_t; dy) = \prod_{a \in \mathbb{A}} \pi^a(x_t; dy^a),$$

on a subset E_0 of the configuration space $E = C^{\mathbb{A}}$. The individual transition probabilities will have an interactive structure, but the transition to a new configuration is made independently by different agents. We consider the case where the influence of the configuration x on the agent a is felt through the local situation $(x^b)_{b \in N(a)}$ is some *neighborhood* $N(a)$ and through a signal about the average situation throughout the entire population.

Introducing the notion of local interactions requires to endow the countable set $\mathbb{A}$ with the structure of a graph where the agents are the knots and where interactive links between certain pairs of agents exist. In view of the global component in the agents' choice probabilities we restrict ourselves to the case $\mathbb{A} := \mathbb{Z}^d := \{a = (a_1, \ldots, a_d) : a_k \in \mathbb{N}\}$ where the agents are located on the d-dimensional integer lattice. The reference groups take the form

$$N(a) := \{b \in \mathbb{Z}^d : \max_k |b_k - a_k| \le l\} \quad \text{for some } l \in \mathbb{N}.$$

In terms of the peer groups $N(a)$ we can model situations where the agents' expectations depend on the previous benchmarks of some acquaintances.[2]

[2] Note that an individual agent affects the next state of just 2^{ld} other traders. Hence no individual person is able to affect the whole market in one single period.

But the behavior of traders also depends on the their information about the prevailing market mood. We consider the simplest case where the agents observe a common random signal $s \in S := \{s^1, \dots, s^M\}$ about aggregate behavior. The conditional probability $\pi_s^a(x; c)$ that the new state of the agent $a \in \mathbb{A}$ is $c \in C$, given the signal s and the configuration x_t, is described in terms of a family of stochastic kernels π_s^a from E to C.

Before the specify the agents' transition probabilities, we illustrate our notion of local and global interactions by means of the following example.

Example 2.4 *Let $C = \{0, 1\}$ and assume that the empirical average*

$$m(x) := \lim_{n\to\infty} \frac{1}{|\mathbb{A}_n|} \sum_{a\in\mathbb{A}_n} x^a$$

associated to $x \in E$ exists. We specify the transition probability of the agent $a \in \mathbb{A}$, given the signal $s \in S \subset [0, 1]$ in terms of a convex combination,

$$\pi_s^a(x; 1) = \gamma_1 x^a + \gamma_2 m^a(x) + \gamma_3 s, \tag{9}$$

of his current state, the proportion $m^a(x)$ of '1' in his neighborhood $N(a)$, and the signal s about the average $m(x)$. For a fixed process $\{s_t\}_{t\in\mathbb{N}}$ the law of large numbers shows that almost surely

$$m(x_{t+1}) = \gamma_1 m(x_t) + \gamma_2 m(x_t) + \gamma_3 s_t.$$

Thus, the sequence of empirical averages $\{m(x_t)\}_{t\in\mathbb{N}}$ may be viewed as a Markov chain on the state space $[0, 1]$ if $s_t \sim Q(m_t; \cdot)$.

The interaction between different agents is homogeneous in that all agents react in the same manner to the states of neighbors and to the signal about aggregate behavior. In order to make this more precise, we introduce the shift maps θ_a on E by $(\theta_a x)(b) = x^{a+b}$.

Assumption 2.5 *For any $s \in S$, there is a stochastic kernel π_s such that*

$$\pi_s^a(x; c) = \pi_s(\theta_a x; c) \quad \textit{for all } c \in C. \tag{10}$$

The probabilities $\pi_s(x; \cdot)$ depend continuously on s, and

$$\pi_s(\theta_a x; \cdot) = \pi_s(\theta_a y; \cdot) \quad \textit{if } \theta_a x = \theta_a y \textit{ on } N(a).$$

We are now ready to specify the conditional distribution of the new states, given a signal about average behavior. For a fixed pair (x_t, s_t), the distribution of the new configuration takes the product form

$$\Pi_{s_t}(x_t; \cdot) := \prod_{a\in\mathbb{A}} \pi_{s_t}(\theta_a x_t; \cdot). \tag{11}$$

The full dynamics of the microscopic process along with the dynamics of aggregate behavior is described in the next section.[3]

[3]Since our focus is on analyzing the impact of contagion and imitation effects on the

2.3 The dynamics of aggregate behavior

Due to the local dependence of the individual transition laws on the current configuration, the dynamics of the mood of the market typically cannot be described by a Markov chain. In order to analyze the asymptotics of both aggregate behavior and asset prices we need a more general mathematical framework which allows us to study convergence properties of locally and globally interacting Markov chains on infinite product spaces. Such a framework has recently been developed by Föllmer and Horst (2001).[4]

Definition 2.6 *A probability measure μ on E is called ergodic, if it is invariant under the shift maps θ_a, i.e., if $\mu = \mu \circ \theta_a$, and if it satisfies a 0-1-law on the σ-field of all shift invariant events.*

We denote the class of all ergodic probabilities on E by $\mathcal{M}_0(E)$. For $n \in \mathbb{N}$, we put $\mathbb{A}_n := [-n,n]^d \cap \mathbb{A}$, and E_0 is the set of all configurations $x \in E$ such that the *empirical field* $R(x)$, defined as the weak limit

$$R(x) := \lim_{n \to \infty} \frac{1}{|\mathbb{A}_n|} \sum_{a \in \mathbb{A}_n} \delta_{\theta_a x}(\cdot),$$

exists and belongs to $\mathcal{M}_0(E)$. The empirical field $R(x)$ carries all *macroscopic information* contained in the configuration $x \in E_0$. The empirical distribution $\varrho(x)$ defined in (5), for instance, is given as the one-dimensional marginal distribution of $R(x)$, and the average action $m(x)$ is given by

$$m(x) = \int_E y^0 R(x)(dy).$$

Assumption 2.7 *The conditional law $Q(R(x); \cdot)$ of the signal $s \in S$ given the empirical field $R(x)$ is specified in terms of a signal kernel Q from $\mathcal{M}_0(E)$ to the finite signal space S. The kernel Q satisfies the Lipschitz condition*

$$\sup_{s, R \neq \hat{R}} \frac{|Q(R;s) - Q(\hat{R};s)|}{d(R,\hat{R})} < \infty \tag{12}$$

with respect to some metric d that induces the weak topology on $\mathcal{M}_0(E)$, and

$$\inf_{R,s} Q(R,s) > 0. \tag{13}$$

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formation of stock prices we do not allow for a feedback from past prices into the behavior of agents. Feedbacks as well as elements of forward looking behavior ("rational expectations") are left for future research.

[4] All results in this section are stated without proofs. For details we refer to reader to Föllmer and Horst (2001) or Horst (2002) and references therein.

Remark 2.8 *Our Assumption 2.7 excludes the case where the agents have complete information about the mood of the market. This assumption is justified if we think of the traders as being small investors. Mathematically, condition (13) allows us to prove a convergence result for the mood of the market without any restrictions on the dependence of the individual transition laws on the signal about aggregate behavior.*

We are now in a position to describe the dynamics of both individual and aggregate behavior in our financial market model. The conditional transition probabilities Π_s introduced in (11) along with the kernel Q determine the transition probability of the microscopic process as

$$\Pi(x;\cdot) := \int_S \Pi_s(x;\cdot)Q(R(x);ds) \quad \text{for } x \in E_0. \tag{14}$$

By Proposition 3.2 in Föllmer and Horst (2001), we have $\Pi_s(x;E_0) = 1$ if $x \in E_0$, and so Π can be viewed as a stochastic kernel on the state space E_0. The empirical field $R(y)$ exists $\Pi(x;\cdot)$-almost-surely and takes the form

$$R(y) = u(R(x), s) := \int_E \Pi_s(y;\cdot)R(x)(dy). \tag{15}$$

We choose E_0 as the state space of our *microscopic process*, and denote by $\mathbb{P}_x$ the distribution of the Markov chain $\{x_t\}_{t\in\mathbb{N}}$ with start in $x \in E_0$. The process $\{x_t\}_{t\in\mathbb{N}}$ induces $\mathbb{P}_x$-a.s. the *macroscopic process* $\{R(x_t)\}_{t\in\mathbb{N}}$. The next theorem shows that the latter sequence may be regarded as a Markov chain on the state space $\mathcal{M}_0(E)$.[5]

Theorem 2.9 *Under $\mathbb{P}_x$ ($x \in E_0$) the macroscopic process is a Markov chain on $\mathcal{M}_0(E)$ with initial value $R(x)$. Is transition operator U acts on bounded measurable functions $f : \mathcal{M}_0(E) \to \mathbb{R}$ according to*

$$Uf(R(x)) = \int f \circ u(R(x), s)Q(R(x);ds)$$

where the map $u : \mathcal{M}_0(E) \times S \to \mathcal{M}_0(E)$ is defined in (15).

Our aim is to show that the mood of the market converges in distribution if the interaction between different agents is not too strong. In order to specify a suitable notion of weak interaction we introduce vectors $r^s = (r^s_a)_{a\in\mathbb{A}}$ ($s \in S$) with components

$$r^s_a = \frac{1}{2}\sup\left\{|\pi_s(x;c) - \pi_s(y;c)| : x^b = y^b \text{ for all } b \neq a, c \in C\right\}.$$

[5] Due to the local dependence of the agents' transition laws on the current configuration, we can typically not expect a Markov property on the level of empirical distributions. This motivated and justifies our general mathematical framework.

The quantity r_a^s measures the dependence of the new state of agent 0 on the current state of agent a, given the signal s. Asymptotic stationarity of the mood of the market can now be guaranteed by limiting the strength of interactions between different agents. More precisely, we assume that the following condition is satisfied.

Assumption 2.10

$$\alpha := \sup_s \sum_a r_{a,0}^s < 1 \tag{16}$$

The following simple example illustrates our weak interaction condition.

Example 2.11 *Let us return to the situation analyzed in Example 2.3 and assume that the individual transition probabilities,*

$$\pi_s^a(x; 1) = g_s\left(\{x^b\}_{b \in N(a)}\right),$$

are described in terms of differentiable maps $g_s : C^{|N(a)|} \to [0, 1]$. *If we denote the partial derivative of* g_s *with respect to* x^b *by* g_s^b, *then our weak interaction condition is satisfied if*

$$\max_{s \in S} \sum_{a \in N(0)} \max_{x^b \in C} \left| g_s^a(\{x^b\}_{b \in N(0)}) \right| < 1.$$

We are now ready to state the main result of this section. Its proof is given in Horst (2002).

Theorem 2.12 *Under Assumptions 2.5, 2.7, and 2.10 the following holds:*

(i) There exists a unique probability measure $\mathbb{Q}^*$ *on the canonical path space of the microscopic process such that the sequence* $\{R(x_t)\}_{t \in \mathbb{N}}$ *is stationary and ergodic under* $\mathbb{Q}^*$.

(ii) Independently of the initial configuration $x \in E_0$, *the macroscopic process* $\{R(x_t)\}_{t \in \mathbb{N}}$ *converges in law to a unique limiting distribution.*

So far, we formulated conditions on the behavior of individual agents which guarantee that the mood of the market settles down in the long run. In the following section we apply this result in order to establish convergence properties of the induced stock price process.

3 Dynamics of the stock price process

In our financial market model the price fluctuations can be highly volatile in periods where the effect of trend chasing becomes too strong. In this section we show that the overall behavior of the price process is nevertheless ergodic if the destabilizing effects of chartists are on average not too strong.

If the environment for the evolution of stock prices is already in equilibrium, i.e., if the sequence $\{\tilde{\varrho}_t\}_{t\in\mathbb{N}}$ is ergodic, then the asymptotic behavior of the price process can be analyzed using methods and techniques from Brandt (1986) or Borovkov (1998). Economically, however, a stationarity condition on the mood of the market is rather restrictive. On the other hand, we derived conditions on the behavior of individual agents which guarantee that the macroscopic process settles down in the long run. Now, our goal is to show that asymptotic stationarity of the driving sequence is enough to guarantee long run stability of the asset price process if the impact of trend chasing is on average not too strong.

3.1 The discrete-time stock price process

Our stability result for the asset price process will be based on a convergence theorem for linear stochastic difference equations in the non-stationary random environments. In order to apply Theorem 2.2 in Horst (2001) we need to show that the environment for the evolution of the price process has a nice tail structure in the sense of the following definition.

Definition 3.1 *Let $\psi := \{(A_t, B_t)\}_{t\in\mathbb{N}}$ is a sequence of $\mathbb{R}^2$-valued random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\hat{\mathcal{F}}_t := \sigma(\{\psi_t\}_{s\geq t})$ be the σ-field generated by the random variables ψ_s for $s \geq t$, and denote by*

$$\mathcal{T}_\psi := \bigcap_{t\in\mathbb{N}} \hat{\mathcal{F}}_t, \tag{17}$$

be the tail-σ-algebra generated by ψ. We say that ψ has a nice tail structure *with respect to a probability measure $\mathbb{Q}$ on $(\Omega, \mathcal{F})$ if the following holds:*

(i) ψ is stationary and ergodic under $\mathbb{Q}$ and satisfies

$$\mathbb{E}_\mathbb{Q} \log|A_0| < 0 \quad \text{and} \quad \mathbb{E}_\mathbb{Q}(\log|B_0|)^+ < \infty \tag{18}$$

where $\mathbb{E}_\mathbb{Q}$ denotes the expectation with respect to the measure $\mathbb{Q}$.

(ii) The asymptotic behavior of ψ is the same under $\mathbb{P}$ and $\mathbb{Q}$, i.e.,

$$\mathbb{P} = \mathbb{Q} \quad \text{on} \quad \mathcal{T}_\psi. \tag{19}$$

Continuity of the total variation distance $\|\cdot\|$ along increasing and decreasing σ-algebras yields (Föllmer (1979), Remark 2.1)

$$\lim_{t\to\infty} \|\mathbb{P} - \mathbb{Q}\|_{\hat{\mathcal{F}}_t} = \|\mathbb{P} - \mathbb{Q}\|_{\mathcal{T}_\psi}. \tag{20}$$

Hence, a sequence ψ satisfies (19) if and only if it becomes stationary in the long run. Under a mild technical condition on the kernel $\tilde{Q}$ introduced in (6), the latter condition allows us to show that the driving sequence

$$\hat{\psi} := \{(f(\tilde{\varrho}_t), g(\tilde{\varrho}_t))\}_{t\in\mathbb{N}} \tag{21}$$

for the price process has a nice tail structure with respect to the unique limiting measure $\mathbb{Q}^*$ of the macroscopic process.

Remark 3.2 *Under the assumptions of Theorem 2.12, the sequence of empirical distributions $\{\varrho(x_t)\}_{t\in\mathbb{N}}$ converges in law to a unique limiting measure. This, however, does not guarantee that the process $\{\varrho(x_t)\}_{t\in\mathbb{N}}$ itself has a nice asymptotic behavior. For this reason we assume that the set of agents who are directly involved in the formation of stock prices is a "representative" subset of the larger set $\mathbb{A}$ of traders.*

The following technical results appear as Corollary 3.30 and Lemma 4.32, respectively, in Horst (2000).

Lemma 3.3 *For $x \in \mathbb{E}_0$, let $\mathbb{P}_x$ be the probability measure on the canonical path space of the microscopic process such that $\mathbb{P}_x[x_0 = x] = 1$. If the stochastic kernel $\tilde{Q}$ from $\mathcal{M}_0(E)$ to $\mathcal{M}(C)$ satisfies the Lipschitz condition*

$$\sup_{s\in S} |\tilde{Q}(R; s) - \tilde{Q}(\hat{R}; s)| \leq Ld(R, \hat{R}), \tag{22}$$

similar to (12), then the following holds:

(i) The sequence $\hat{\psi}$ has a nice tail structure with respect to $\mathbb{Q}^*$.

(ii) The sequence $\hat{\psi}$ is φ-mixing under both $\mathbb{Q}^*$ and $\mathbb{P}_x$, and there exists a constant $M < \infty$ such that the n-th mixing coefficient is bounded from above by $M\alpha^n$. Here α is defined in (16).

Due to the first part of Lemma 3.3, the environment for the evolution of the stock price process has a nice asymptotic behavior. Thus, the results in Horst (2001) allow us to introduce a quantitative bound on the aggregate effects of interactions which guarantees that the price process is driven into a stationary regime. Stock prices are stationary in the long run if the mood of the market settles down as $t \to \infty$, and if asymptotically the destabilizing effects of trend chasing are weak enough.

Proposition 3.4 *Suppose that the assumptions of Theorem 2.12 are satisfied. If $\tilde{Q}$ satisfies the Lipschitz condition (22), and if*

$$\mathbb{E}_{\mathbb{Q}^*} \log |f(\tilde{\varrho})| < 0 \quad \textit{and} \quad \mathbb{E}_{\mathbb{Q}^*} (\log |g(\tilde{\varrho}))^+ < \infty, \tag{23}$$

then the price process converges in law to a unique limiting measure.

The first assumption in (23) may be viewed as a mean contraction condition on the environment $\hat{\psi}$. If the destabilizing effects of the environment become too strong, i.e., if the mean contraction condition does not hold, then prices tend to zero or go off to infinity.

3.2 A diffusion approximation for the stock price process

To make the qualitative behavior of the asset price process more transparent we apply in this section an invariance principle to the environment for the evolution of the asset price process. This leads to an approximation of the price process (8) in continuous time. A Similar approach has been taken by Neson (1990), Neson and Ramaswamy (1990) and Föllmer and Schweizer (1993) to obtain diffusion approximations for price processes evolving in an *ergodic* random environment from a sequence of suitably specified discrete-time processes. We extend these results by replacing the stationarity assumption by an asymptotic stability condition on the mood of the market.

The convergence concept we use is weak convergence on the Skorohood space $\mathbb{D}^d$ of all $\mathbb{R}^d$-valued right-continuous functions with left limits on $[0, \infty)$, endowed with the weak topology. Moreover, we denote by $\mathrm{Law}(X, \mathbb{P})$ the distribution of a random variable X under the measure $\mathbb{P}$, and $\stackrel{w}{\longrightarrow}$ means weak convergence of probability measures, and .

3.2.1 A Central Limit Theorem for Non-Stationary Sequences

The proof of our approximation result is based on a diffusion approximation for the discrete-time linear stochastic difference equation

$$P_{t+1} - P_t = A_t P_t + B_t \qquad (t \in \mathbb{N})$$

environment $\psi = \{(A_t, B_t)\}_{t \in \mathbb{N}}$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We assume that ψ is nice with respect to some measure $\mathbb{Q}$ on $(\Omega, \mathcal{F})$ and introduce discrete-time processes $P^n = \{P_t^n\}_{t \in \mathbb{N}}$ by

$$P_{t+1}^n - P_t^n = \frac{1}{\sqrt{n}} A_t P_t^n + \frac{1}{\sqrt{n}} B_t. \tag{24}$$

We identify P^n with the continuous-time process $(P_{[nt]}^n)_{t \geq 0}$. In terms of the quantities

$$X_t^n := \frac{1}{\sqrt{n}} \sum_{i=0}^{[nt]} A_i \quad \text{and} \quad Y_t^n := \frac{1}{\sqrt{n}} \sum_{i=0}^{[nt]} B_i, \tag{25}$$

equation (24) translates into the stochastic differential equation

$$dP_t^n = P_{t-}^n dX_t^n + dY_t^n. \tag{26}$$

Let us first consider the benchmark case where $\mathbb{P} = \mathbb{Q}$, i.e., the case where ψ is stationary and ergodic. To this end, we denote by $W = (W_1, W_2)$ a two-dimensional Brownian motion defined on $(\Omega, \mathcal{F}, \mathbb{P})$. Under standard assumptions on the environment ψ given in, e.g., Billingsley (1968),

$$\mathrm{Law}(Z^n, \mathbb{Q}) \stackrel{w}{\longrightarrow} \mathrm{Law}(V \cdot W, \mathbb{Q}) \quad \text{where} \quad Z^n := (X^n, Y^n) \tag{27}$$

and V is a suitable 2×2 matrix. If the process $\{Z^n\}_{n\in\mathbb{N}}$ is also "good" in the sense of Definition 4.2 in Duffie and Protter (1992), then (27) implies

$$\mathrm{Law}((Z^n, P^n), \mathbb{Q}) \xrightarrow{w} \mathrm{Law}((V \cdot W, P), \mathbb{Q})$$

where $P = \{P_t\}_{t\geq 0}$ is the unique solution of the stochastic differential equation

$$dP_t = P_t dX_t + dY_t. \tag{28}$$

The solution of (28) may be viewed as an Ornstein-Uhlenbeck process in a random environment. Its qualitative behavior is investigated in Föllmer and Schweizer (1993).

We are now going to prove a functional central limit theorem for diffusion processes in non-stationary random environments.

Proposition 3.5 *Suppose that ψ has a nice tail structure with respect to a measure $\mathbb{Q}$, that $\mathbb{E}_{\mathbb{Q}}A_0 = \mathbb{E}_{\mathbb{Q}}B_0 = 0$, that (27) is satisfied, and that the sequence $\{Z^n\}_{n\in\mathbb{N}}$ is good under the original measure $\mathbb{P}$. Then*

$$\mathrm{Law}((Z^n, P^n), \mathbb{P}) \xrightarrow{w} \mathrm{Law}((V \cdot W, P), \mathbb{Q}).$$

Proof: Let us first show that $\mathrm{Law}(Z^n, \mathbb{P}) \xrightarrow{w} \mathrm{Law}(V \cdot W, \mathbb{Q})$. To this end, we fix an increasing sequence $\{\sigma_n\}_{n\in\mathbb{N}}$ such that $\sigma_n/\sqrt{n} \to 0$ as $n \to \infty$. For a given time horizon $T > 0$, and for each $n \in \mathbb{N}$, we introduce the two-dimensional process $\{\widetilde{Z}_t^n\}_{0\leq t\leq T}$ by

$$\widetilde{Z}_t^n := \begin{cases} \frac{1}{\sqrt{n}} \sum_{i=\sigma_n}^{[nt]} (A_i, B_i) & \text{if } \frac{\sigma_n}{\sqrt{n}} \leq t \leq T \\ 0 & \text{otherwise.} \end{cases}$$

We denote by $d_0(\cdot,\cdot)$ and $\mathcal{B}_{\mathbb{D}}$ the Skorohood metric[6] and the Borel-σ-field on the space $\mathbb{D}_{\mathbb{R}^2}[0,T]$, respectively. Then

$$d_0(Z^n, \widetilde{Z}^n) \leq \frac{\sigma_n}{\sqrt{n}} \left| \left(\frac{1}{\sigma_n} \sum_{i=0}^{\sigma_n} |A_i|, \frac{1}{\sigma_n} \sum_{i=0}^{\sigma_n} |B_i| \right) \right|. \tag{29}$$

Since $\mathbb{P} = \mathbb{Q}$ on the tail-field generated by the sequence ψ and because ψ is ergodic under $\mathbb{Q}$, the series

$$\frac{1}{\sigma_n} \sum_{i=0}^{\sigma_n} |A_i| \quad \text{and} \quad \frac{1}{\sigma_n} \sum_{i=0}^{\sigma_n} |B_i|$$

are $\mathbb{P}$- and $\mathbb{Q}$-almost surely convergent, and $\lim_{n\to\infty} \frac{\sigma_n}{\sqrt{n}} = 0$ yields

$$\lim_{n\to\infty} d_0(Z^n, \widetilde{Z}^n) = 0 \qquad \mathbb{P}\text{-a.s. and } \mathbb{Q}\text{-a.s.} \tag{30}$$

[6] For the definition of d_0 see, e.g., Billingsley (1968), p. 113.

Since the event $\{\widetilde{Z}^n \in B\}$ ($B \in \mathcal{B}_{\mathbb{D}}$) belongs to the σ-algebra $\hat{\mathcal{F}}_{\sigma_n}$ and because ψ has a nice tail structure there exists a decreasing sequence $\{c_n\}_{n \in \mathbb{N}}$ that satisfies

$$\sup_B \left| \mathbb{P}[\widetilde{Z}^n \in B] - \mathbb{P}^*[\widetilde{Z}^n \in B] \right| \leq c_n. \tag{31}$$

Let us now denote by Q^* the law of the Gaussian martingale $V \cdot W$ under the measure $\mathbb{Q}$ and fix a Q^*-continuous set $B \in \mathcal{B}_{\mathbb{D}}$. Since

$$\lim_{n \to \infty} \mathbb{Q}[Z^n \in B] = Q^*[B]$$

equation (30) along with Theorem 4.2 in Billingsley (1968) yields

$$\lim_{n \to \infty} \mathbb{Q}[\widetilde{Z}^n \in B] = Q^*[B].$$

Using (31) we see that

$$\lim_{n \to \infty} \mathbb{P}[\widetilde{Z}^n \in B] = Q^*[B].$$

Therefore, (30) and Theorem 4.2 in Billingsley (1968) imply that

$$\text{Law}(Z^n, \mathbb{P}) \xrightarrow{w} \text{Law}(V \cdot W, \mathbb{Q}).$$

Hence the assertion follows from the goodness property of the sequence $\{Z^n\}_{n \in \mathbb{N}}$ under $\mathbb{P}$. □

3.2.2 An approximation result for the price process

We prove our approximation result for the price process under the additional assumption that the environment for the evolution of asset prices is asymptotically described by mean-zero stochastic processes. This assumption can be relaxed; for details we refer the reader to Chapter 4 in Horst (2000).

Assumption 3.6 *the assumptions of Theorem 2.12 are satisfied with bounded functions $f, g : \mathcal{M}(C) \to \mathbb{R}$, and under the measure $\mathbb{Q}^*$, we have that*

$$\mathbb{E}_{\mathbb{Q}^*} f(\tilde{\varrho}) = 1, \quad \mathbb{E}_{\mathbb{Q}^*} g(\tilde{\varrho}) = 0, \quad \mathbb{E}^2_{\mathbb{Q}^*} f(\tilde{\varrho}) < \infty, \quad \mathbb{E}^2_{\mathbb{Q}^*} g(\tilde{\varrho}) < \infty.$$

We are now ready to show how our sequence of temporary price equilibria $\{p_t\}_{t \in \mathbb{N}}$ can be approximated in law by a continuous-time process $(P_t)_{t \geq 0}$ if the mood of the market settles down in the long run.

Theorem 3.7 *If Assumption 3.6 holds and if the kernel $\tilde{Q}$ defined in (6) satisfies the Lipschitz condition (22), then the logarithmic price process can be approximated in law by a continuous-time process of the form (28).*

Proof: In the stationary setting, i.e., under the measure $\mathbb{Q}^*$, an invariance principle can be applied to the sequence

$$X_t^n := \frac{1}{\sqrt{n}} \sum_{i=0}^{[nt]} (f(\tilde{\varrho}_i) - 1) \quad \text{and} \quad Y_t^n := \frac{1}{\sqrt{n}} \sum_{i=0}^{[nt]} g(\tilde{\varrho}_i),$$

due to Lemma 3.3 (ii) and Billingsley (1968). Thus, our assertion follows from Proposition 3.5 if we can show that and that for any $x \in E_0$ the sequence $\{Z^n\}_{n\in\mathbb{N}} = \{(X^n, Y^n)\}_{n\in\mathbb{N}}$ defined on $(\Omega, \mathcal{F}, \mathbb{P}_x)$ is good. To this end, we introduce the σ-fields

$$\mathcal{G}_t := \sigma\left(\{f(\tilde{\varrho}_i), g(\tilde{\varrho}_i)\} : 0 \le i \le t\right) \qquad (t \in \mathbb{N})$$

and processes $M = \{M_t\}_{t\in\mathbb{N}}$ and $A = \{A_t\}_{t\in\mathbb{N}}$ by

$$M_t := \begin{pmatrix} \sum_{k=0}^t (f(\tilde{\varrho}_k) - 1) + \sum_{k=0}^\infty \hat{\mathbb{E}}_\xi[f(\tilde{\varrho}_{k+t}) - 1|\mathcal{G}_t] \\ \sum_{k=0}^t (g(\tilde{\varrho}_k) - 1) + \sum_{k=0}^\infty \hat{\mathbb{E}}_\xi[g(\tilde{\varrho}_{k+t}) - 1|\mathcal{G}_t] \end{pmatrix} \tag{32}$$

and

$$A_t := \begin{pmatrix} \sum_{k=0}^\infty \hat{\mathbb{E}}_\xi[f(\tilde{\varrho}_{k+t}) - 1|\mathcal{G}_t] \\ \sum_{k=0}^\infty \hat{\mathbb{E}}_\xi[g(\tilde{\varrho}_{k+t}) - 1|\mathcal{G}_t] \end{pmatrix}. \tag{33}$$

By Lemma 3.3 (ii), the environment $\{\tilde{\varrho}_t\}_{t\in\mathbb{N}}$ is φ-mixing under $\mathbb{P}_x$, and the n-th mixing coefficient is bounded above by $M\alpha^n$. Thus, the series in (32) and (33) are almost surely absolutely convergent; see, e.g., Ethier and Kurtz (1986). Furthermore, M is a vector of square integrable martingales with respect to the measure $\mathbb{P}_x$ and the filtration $\{\mathcal{G}_t\}_{t\in\mathbb{N}}$. In terms of the quantities $M^n = \{M_{[nt]}\}_{t\ge 0}$ and $A^n = \{A_{[nt]}\}_{t\ge 0}$ we have

$$Z_t^n = \frac{1}{\sqrt{n}} M_t^n - \frac{1}{\sqrt{n}} A_t^n.$$

Since the martingales M^n have uniformly bounded expected jumps, it follows from Theorem 4.3 in Duffie and Protter (1992), that the sequence $\{Z^n\}_{n\in\mathbb{N}}$ is good if

$$\sup_{n\in\mathbb{N}} \{\mathbb{E}_x[|A^n|_T]\} < \infty$$

where $|A_T^n|$ denotes the total variation of the process A^n on the time interval $[0, T]$. This, however, follows from standard estimates as in, e.g., Duffie and Protter (1992), Example 6.3. □

Extending a result of Brandt (1986) from discrete to continuous time, Föllmer and Schweizer (1993) proved that logarithmic price converges almost surely to an ergodic process $(\hat{P}_t)_{t\ge 0}$ in the sense that

$$\lim_{t\to\infty} |P_t - \hat{P}_t| = 0 \qquad \mathbb{P}\text{-a.s.}$$

In particular, the price process converges in distribution and it turns out that the invariant distribution can be given in closed form; see Chapter 4 in Föllmer and Schweizer (1993).

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