Stochastic Cascades, Credit Contagion, and Large Portfolio Losses: Internet Supplement

Ulrich Horst
University of British Columbia

14th February 2005

Abstract

This internet supplement summarizes some mathematical results on which the results of the original paper are based.

1 Poisson distribution, branching processes and random sums

This mathematical appendix to our paper on "Stochastic Cascades, Credit Contagion, and Large Portfolio Losses" summarizes properties of the Poisson distribution, branching processes and simple compound distributions. Proofs can be found in, e.g., Resnick (1992), Chapter 1.

1.1 The Poisson distribution

A integer-valued random variable X defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is Poisson distributed with parameter ν, $X \sim \mathcal{P}(\nu)$, if

$$\mathbb{P}[X = k] = \pi_k(\nu) = \frac{\nu^k e^{-\nu}}{k!}. \tag{1}$$

Its moment generating function $F(x) = \sum_{k \geq 0} \pi_k(\nu) s^k$ takes the form

$$F(x) = e^{\nu(x-1)}. \tag{2}$$

If $X_1 \sim \mathcal{P}(\nu_i)$ $(i = 1, 2)$ are independent, then $X_1 + X_2 \sim \mathcal{P}(\nu_1 + \nu_2)$.

The Poisson distribution can be viewed as an approximation of the binomial distribution for large n and small success probabilities. More specifically, let $b(n, p)$ be the distribution of the number of successes in n Binomial trails when the success probability is p. If $X_n \sim b(n, p(n))$ and

$$\lim_{n\to\infty} np(n) = \lim_{n\to\infty} \mathbb{E}X_n = \nu \in (0, \infty),$$

then the sequence $\{X_n\}$ converges in distribution to a random variable X where $X \sim \mathcal{P}(\nu)$; see, e.g., Resnick (1992), p. 29. This property of the binomial distribution is in fact the key to the proof of our Theorem 2.7.

1.2 Compound Poisson distributions

Let $(Z_i)_{i\in\mathbb{N}}$ be a sequence of independent and identically distributed random variables with moment generating function F, and let N be independent of $(Z_i)_{i\in\mathbb{N}}$ and Poisson distributed with parameter λ. The random sum

$$S = Z_1 + \cdots + Z_N$$

follows a *compound Poisson distribution*. Its moment generating function takes the form

$$G(x) = \exp\left(\lambda(F(x) - 1\right)$$

and so the mean and variance of S_N are given by, respectively,

$$\mathbb{E}S = \lambda \mathbb{E}Z_1 \qquad \text{and} \qquad \mathbb{V}S = \lambda \left(\mathbb{V}Z_1 + (\mathbb{E}Z_1)^2\right),$$

Remark 1.1 *If the compounding variables Z_i are distributed according to a Borel Tanner distribution with parameter ν, then an application of Lagrange's theorem on the inversion of series yields*

$$\mathbb{E}Z_1 = \frac{1}{1-\alpha} \qquad \textit{and} \qquad \mathbb{V}Z_1 = \frac{\alpha}{(1-\alpha)^3}.$$

In this case the mean and variance of the random sum S take the respective forms

$$\mathbb{E}S = \frac{\lambda}{1-\nu} \qquad \textit{and} \qquad \mathbb{V}S = \frac{\lambda}{(1-\nu)^3}.$$

A random variable X has a geometric distribution if for $k = 0, 1, 2, \ldots$

$$\mathbb{P}[X = k] = (1-p)^k p \qquad (0 \le p \le 1)$$

which is the distribution of the number of failures before the first success in repeated Bernoulli trials. The moment generating function takes the form

$$F(x) = \frac{p}{1-qx} \qquad \text{for} \qquad 0 < x < \frac{1}{q}.$$

If $\{\hat{Z}_t\}_{t\in\mathbb{N}}$ is a sequence of independent geometrically distributed random variable and if $Z_t := 1 + \hat{Z}$, then $S = Z_1 + \cdots + Z_N$ has moment generating function

$$G(x) = \exp\left(\lambda\left[\frac{px}{1-(1-p)x} - 1\right]\right) \qquad \text{for} \qquad N \sim \mathcal{P}(\lambda).$$

1.3 Branching processes

The following definition recalls the notion of a branching process. For a detailed discussion of branching processes we refer the interested reader to Harris (1989).

Definition 1.2 *Let $\{Z_{n,j}, n, j \in \mathbb{N}\}$ be a family of independent and identically distributed random variables defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The sequence $\{Z_n\}$ defined recursively by*

$$Z_0 = l \quad \textit{and} \quad Z_n = Z_{n,1} + \cdots + Z_{n,Z_{n-1}} \tag{3}$$

is called a branching process with l sister ancestors.

The quantity Z_n can be thought of as the size of a population starting out with a single ancestor. The random variable $Z_{n,j}$ describes the number of members of the n-th generation which are offsprings of the j-th member of the $(n-1)$-st generation, and the initial population is size l. Clearly, the state 0 is an absorbing state, and we denote by

$$\tau := \inf\{n : Z_n = 0\}$$

the time of extinction. It is well known that a simple branching process exhibits an instability: either extinction occurs or the process explodes with positive probability. More specifically

$$\mathbb{P}[\tau < \infty] = 1 - \mathbb{P}[Z_n \uparrow \infty].$$

The following theorem states that a population dies out if, on average, each particle produces at most one descendent.

Theorem 1.3 *Let $\{Z_n\}$ be a branching process in the sense of Definition 1.2. If a parent generation produces on average at most one offspring, i.e., if*

$$\mathbb{E}[Z_{n,j}] \leq 1$$

then the population dies out almost surely:

$$\mathbb{P}[\tau < \infty] = 1.$$

If $\{Z_n\}$ is a branching process with an almost surely finite time of extinction, then the total progeny

$$Z := l + Z_1 + Z_2 + \cdots + Z_\tau$$

is finite with probability 1. In this case the random variables Z_k are independent, with the same distribution as Z. The tail of the distribution of Z can be specified in terms of the moment generating function $F : (0, \beta) \to \mathbb{R}$ of $Z_{n,j}$, due to a seminal result by Otter (1949). If there exists $x^* \in (0, \beta^*)$ that satisfies

$$F'(x^*) = \frac{F(x^*)}{x^*} \tag{4}$$

then there is a multiplicative constant $C < \infty$ such that

$$\mathbb{P}[Z = k] = Cr^{-k-\frac{1}{2}} k^{-\frac{3}{2}} \qquad \text{as} \qquad k \to \infty \tag{5}$$

where $r := \frac{x^*}{F(x^*)}$; see also Harris (1989), Theorem I.13.1. For the special case where

$$Z_{n,j} \quad \text{is Poisson distributed with parameter } \nu,$$

i.e., for a population that reproduces from generation to generation in a Poisson way,

$$\mathbb{P}[\tau < \infty] \quad \text{if and only if} \quad \nu \leq 1.$$

In such a situation Z_n is conditionally Poisson distributed given Z_{n-1} with parameter νZ_{n-1}, and the total number of offspring is known to follow a *Borel-Tanner distribution*. For a proof of the following result, we refer the reader to Kingman (1993).

Theorem 1.4 *If the random variables $Z_{n,j}$ follows a Poisson distribution with parameter $\nu \leq 1$, and if $Z_0 = l$, then distribution of the total progeny satisfies*

$$\mathbb{P}[Z = k | Z_0 = l] = \frac{l}{k} \pi_{k-l}(k\nu) = \frac{l}{k} \frac{(k\nu)^{k-l} e^{-k\nu}}{(k-l)!} \quad \text{for } k = l, l+1, \ldots. \tag{6}$$

In particular, for $l = 1$ the random variable Z follows a Borel Tanner distribution. If $Z_0 \sim \mathcal{P}(\mu)$, then

$$Z \stackrel{\mathbb{D}}{=} \sum_{t=1}^{Z_0} Z_t$$

for a sequence $\{Z_t\}_{t \in \mathbb{N}}$ of independent random variables following a Borel-Tanner distribution.

If $Z_{n,j} \sim \mathcal{P}(\nu)$, then the moment generating function is given by (2), and $x^* := \nu^{-1}$ satisfies (4). In this case $\frac{x^*}{F(x^*)} = \frac{1}{\nu e^{1-\nu}}$, and we obtain

$$\mathbb{P}[Z = k | Z_0] = C(\nu e^{1-\nu})^k k^{-\frac{3}{2}} \qquad \text{as} \qquad k \to \infty.$$

Thus, for the limiting case $\nu = 1$, the total population size has a power law distribution.

Corollary 1.5 *If $\nu = 1$, then*

$$\mathbb{P}[Z = k] = Ck^{-\frac{3}{2}} \qquad \textit{as} \qquad k \to \infty. \tag{7}$$

References

Harris, T.E., 1989. The Theory of Branching Processes. Springer-Verlag, Berlin.

Kingman, J.F.C., 1993. Poisson Processes. Oxford University Press, Oxford.

Otter, R., 1949. The multiplicative process, Annals of Mathematical Statistics 20, 206–224.

Resnick, S.I., 1992. Adventures in Stochastic Processes. Birkhäuser, Boston.