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Multiple Markets

New Perspective on Nonlinear Pricing

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Abstract We discuss how linear equilibrium pricing in certain competitive market structures may represent nonlinear equilibrium pricing of *Aliprantis et al.* (2001, 2005). Their work extends the theory of value beyond the scope of the Walrasian single market linear price model. Our arguments include a new and general result on the existence of linear price equilibrium with multiple markets. Each market has its own price vector (linear functional), and agents' involvement in various markets is heterogeneous. As a result, price differences across markets may prevail in equilibrium. We present an example in which single market linear price equilibrium does not exist but certain corresponding equilibrium with two markets does. This example is a particular instance of a prevalent nonexistence problem in atomless economies with differential information. Bypassing the nonexistence problem is one of the achievements of the nonlinear equilibrium theory. Our equilibrium with multiple markets, on the other hand, offers a solution with a more standard economic interpretation. Besides, our general framework is a model of multiple markets in their own right, and our results are related to the role of economic intermediation and bilateral trade.

The idea of this paper occurred to the author while he studied examples of nonlinear price decentralization of Pareto efficient allocations in *Tourky and Yannelis* (2003). The author thanks his PhD supervisor at the ANU, Professor Rabee Tourky, for kindly sharing this working paper, helpful comments, and discussions. Two anonymous referees and the editor are gratefully acknowledged for questions that led to significant improvements. The author appreciates valuable feedback at various stages from Patrick Beissner, Simon Grant, Tai-Wei Hu, M. Ali Khan, Jeff Kline, Kieron Meagher, Idione Meneghel, Romans Pancs, Martin Richardson, Ronald Stauber, Jan Werner, Nicholas Yannelis, Valentin Zelenyuk, and an ANU audience.

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1 Introduction

Even though information asymmetries and limited market participation conceal arbitrage opportunities, economists often seek solution concepts with laws of one price. Not surprisingly, working to ensure that such a solution concept was nonvacuous led *Aliprantis et al.* (2001, 2005) to allowing for nonlinear equilibrium pricing. Their model is useful as it extends the theory of value to markets with missing trade in some options and to infinite dimensional economies with heterogeneous consumption sets. This pertains to limited market participation, which can arise when agents lack information, as in the differential information economies introduced by *Radner* (1968). Infinite dimensional development of this finite dimensional prototype lagged behind the benchmark common knowledge theory of value, particularly with linear pricing. Only forty years later did *Podczeck and Yannelis* (2008) allow for infinitely many state independent commodities but retain the assumption of a finite state space. Only slightly has this assumption been relaxed since then. Such a progress is made by *Hervés-Beloso et al.* (2009), but restricting all information asymmetries to the presence of private signals with only finitely many values. Besides, *Klishchuk* (2015) covers situations where agents have independent information, and also where the total endowment of the economy is common knowledge. Despite these limited equilibrium existence results, the nonexistence is prevalent in the sense of *Anderson and Zame* (2001) according to *Podczeck et al.* (2008). The problem arises because this literature is confined to linear price equilibria in which all agents face the same price vector. This nonexistence problem is distinctive of the Walrasian model and does not affect the solution concept of the core (*Yannelis*, 1991; *Koutsougeras and Yannelis*, 1993). *Aliprantis et al.* (2001) solve this nonexistence problem by allowing for nonlinear pricing, requiring all agents to face the same possibly nonlinear price function.

Our analysis in this paper suggests that the nonlinearities may often be interpreted in terms of linear prices that segment the economy into multiple markets. We define markets by specifying their participants and traded commodity bundles. When the nonlinear price equilibrium is bilaterally feasible, our maximal segmentation delineates pairwise trades and trade-specific prices yielding that equilibrium allocation. These prices are linear but may differ across trades. Such a price disparity echoes the model of the market with pairwise meetings under a different type of asymmetric information developed by *Wolinsky* (1990). “Nonnegligible volumes of trade occur at two different prices” in equilibrium of this model.

Rather than interpreting the nonlinearities solely in terms of pairwise trade, our model is flexible enough to allow for simpler representations if frictions

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are not extreme. Let us consider the case of identical consumption sets, which is frictionless in the sense of full market participation and where Walrasian equilibrium exists in infinite dimensions. In our model of multiple markets, this Walrasian equilibrium can also be represented as an equilibrium of an economy with just a single market. Unlike our maximal segmentation, this single market representation does not specify underlying pairwise trades but is more parsimonious. In intermediate cases, we decompose agents' sales and purchases into marketwise components transacted at the corresponding marketwise prices. Our general framework accommodates all of these segmentation cases in a unified way, and all that changes is the number of markets. While all admissible trades are priced linearly in multiple markets, the induced pricing of commodity bundles agrees with the nonlinear pricing of *Aliprantis et al.* (2001). Our framework can potentially accommodate even more nonlinearities if economic intermediaries are explicitly introduced into the model.

Our equilibrium with multiple markets has a more standard definition than equilibrium with nonlinear prices of *Aliprantis et al.* (2001). They describe a nonlinear price by a family of personalized linear prices indexed by the set of agents. A commodity bundle is priced then at the maximum revenue from distributing the bundle among these individually price-taking agents. But the authors do not explain if such revenue maximization is feasible and satisfies budget constraints given those personalized prices. We replace personalized prices with marketwise prices, introducing carefully chosen market structures which have room for linear pricing (Section 4). We define linear price equilibrium with multiple markets in which individual budget constraints and marketwise feasibility are satisfied (Subsection 4.3). Besides representing nonlinear equilibria as equilibria of economies with multiple markets (Section 5), we give a general equilibrium existence result for such economies (Section 6). An example, where single market linear equilibrium does not exist (Subsection 3.3.3) but equilibrium with two markets does (Subsection 3.3.2), clarifies our ideas. In Section 2, we first further discuss related literature.

2 Further Remarks on Related Literature

Ever since its inception, standard infinite dimensional theory of value has excluded most small consumption sets pertaining to limited market participation. The situation is especially tight with the lattice commodity spaces that motivated *Mas-Colell* (1986) and *Aliprantis and Brown* (1983). Limited market participation translates into heterogeneity of consumption sets and conflicts with the assumption of their equality initiated by *Mas-Colell* (1986). The nonexistence problem arises because this literature is confined to single market linear price equilibria. Such nonexistence is known to be a prevalent situation in atomless economies with differential information (*Tourky and Yannelis*, 2003; *Podczeck et al.*, 2008). We solve this nonexistence problem by modelling those situations as economies with multiple markets in contrast to nonlinear pricing of *Aliprantis et al.* (2001).

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It is important to mention alternative models in which information asymmetries coexist with full market participation. This applies, for instance, to *Correia-da-Silva and Hervés-Beloso* (2009), *Condie and Ganguli* (2011a,b), Definitions 3.5 and 3.6 in *de Castro et al.* (2011), and a special case of *Angelopoulos and Koutsougeras* (2015). In these approaches, agents are to some extent ambiguous about contracts.

Another relevant observation is that the results of *Aliprantis et al.* (2001, 2005) also extend the theory of value beyond vector lattices. Without such structure, the nonexistence problem spreads even to the case of full market participation, as revealed by the example of *Aliprantis et al.* (2004a). Sufficient conditions for the existence of full participation linear price equilibrium in this context are provided by *Aliprantis et al.* (2004b, 2005).

A compelling sufficient condition for the existence of limited participation linear price equilibrium is seen in Remark 9 of *He and Yannelis* (2016). Their condition requires norm compactness of consumption sets, and we note that many sequence spaces admit appealing norm compact consumption sets (see *Wickstead*, 1975).

Finally, let us mention alternative approaches to nonlinear pricing present within the theory of value. They allow dispensing with certain convexity assumptions according to *Chavas and Briec* (2012) and *Habte and Mordukhovich* (2011). The latter paper considers economies with public goods. Nonlinear and linear equilibrium theory for them is studied at great economic generality by *Graziano* (2007).

3 Example

In *Aliprantis et al.* (2001), equilibria with nonlinear prices are called personalized equilibria. This example compares a personalized equilibrium and an equilibrium with multiple markets in an economy where single market linear equilibrium does not exist. To the extent of featuring this nonexistence (Subsection 3.3.3), the example is based on a sufficient condition and other ideas of *Tourky and Yannelis* (2003).

3.1 Economy

We consider a particular differential information economy in the framework of Section 9.2 in *Aliprantis et al.* (2001). Two agents face exogenous uncertainty. It is described by the probability space $([0,1], \Lambda, \lambda)$, where Λ is the σ-algebra of all Lebesgue measurable subsets of $[0,1]$, and λ is the Lebesgue measure on $\mathbb{R}$. Hereinafter, if x is an element of a vector space K and p is a linear functional on K, then the value of p at x is denoted by $p \cdot x$.

Agent 1 has full information represented by the σ-algebra Λ. The agent's consumption set X_1 is the positive cone of the commodity space $L_1([0,1], \Lambda, \lambda)$.

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Agent 2 has a coarser information (σ-algebra) Λ_2 generated by the family of intervals

$$I_n = \left[\frac{n}{n+1}, \frac{n+1}{n+2}\right)$$

with $n \in \{0, 1, 2, \dots\}$. The agent's consumption set X_2 is smaller and limited to Λ_2-measurable nonnegative commodity bundles, i.e.

$$X_2 = \{x \in X_1 : x \text{ is } \Lambda_2\text{-measurable}\}.$$

The description of the economy is completed below in Table 1, where $f_1 : [0,1] \to \mathbb{R}$ is defined by $f_1(s) = 1 - s$.

Table 1 Initial endowments and utility functions for Section 3

Agent	Endowment $\omega_i \in X_i$	Utility function $u_i : X_i \to \mathbb{R}$
1	$\omega_1(s) = s$	$u_1(x) = \int_0^1 f_1(s)\, x(s)\, ds$
2	$\omega_2(s) = 1$	$u_2(x) = \int_0^1 x(s)\, ds$

3.2 Multiple Markets versus Personalized Equilibria

To describe a typical nonlinear price, let $p = (p_1, p_2)$ be a vector of personalized prices both of which are continuous linear functionals on the commodity space. The induced possibly nonlinear price is the real function ψ_p that maps each nonnegative commodity bundle x to its value

$$\psi_p \cdot x = \sup\{p_1 \cdot y_1 + p_2 \cdot y_2 : y \in X_1 \times X_2, y_1 + y_2 \le x\}.$$

Given an $(x_1, x_2) \in X_1 \times X_2$ which is an allocation (satisfies $x_1 + x_2 \le \omega_1 + \omega_2$), the vector (x_1, x_2, p_1, p_2) is said to be a *personalized equilibrium* if

(a) for all i and $x \in X_i$, we have $u_i(x) > u_i(x_i) \implies \psi_p \cdot x > \psi_p \cdot x_i$,
(b) for all $\alpha \in \mathbb{R}^2_+$, we have $\psi_p \cdot (\alpha_1\omega_1 + \alpha_2\omega_2) \le \alpha_1\psi_p \cdot x_1 + \alpha_2\psi_p \cdot x_2$, and
(c) $\psi_p \cdot (\omega_1 + \omega_2) > 0$.

In Subsection 3.3 we find such a personalized equilibrium (x_1, x_2, p_1, p_2), as given in (1) there, and describe it in terms of two markets. Any commodity bundle can be traded in market 1, but only fully informed agents participate in it. By contrast, market 2 accepts only Λ_2-measurable bundles but is open to all agents. Apart from these participation requirements, agents are only constrained by their initial endowments and marketwise prices. We find, in a sense, a market-clearing vector $p' = (p'_1, p'_2) \ne p$ of marketwise prices, which are continuous linear functionals on the commodity space as well. Agents sell their initial endowments in different markets so as to maximize revenue and

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make utility-maximizing purchases in those markets. Agent 1 maximizes revenue by selling a particular Λ_2-measurable bundle $\omega_{12} \in [0, \omega_1]$ in market 2 and the remainder $\omega_{11} = \omega_1 - \omega_{12}$ in market 1. Agent 2 participates and sells the initial endowment ω_2 only in market 2. The corresponding revenues and the marketwise prices determine the agents' budget sets. The consumption x_i of every agent i in the personalized equilibrium turns out to coincide with the agent's total utility-maximizing purchases in different markets. Thus we obtain a competitive equilibrium description of the personalized equilibrium allocation (x_1, x_2).

Notice that $\psi_{p'}$ assigns to each nonnegative commodity bundle the maximum revenue from distributing the bundle among these two markets. Even though below we find $p' \neq p$, they are such that $\psi_{p'} = \psi_p$. Thus we are able to reinterpret nonlinearities of ψ_p in terms of multiple markets.

3.3 Technicalities

Here we confirm our last series of statements as well as the nonexistence of single market linear equilibrium. We view prices p_i and p'_i as elements of $L_\infty\left([0,1], \Lambda, \lambda\right)$ so that corresponding values of a commodity bundle x are

$$p_i \cdot x = \int_0^1 p_i\left(s\right) x\left(s\right) ds$$

and $p'_i \cdot x$ defined analogously. A preliminary step is to decompose ω_1 into

$$\omega_{12} = \sum_{n=0}^{\infty} \frac{n}{n+1} \chi_{I_n} \in X_2 \text{ and } \omega_{11} = \omega_1 - \omega_{12} \in X_1,$$

as illustrated in Figure 1.

3.3.1 Personalized Equilibrium

We show in the next paragraph that a personalized equilibrium is obtained, as illustrated in Figure 2, by posing

$$x_1 = \omega_{11} + \left(4 - \frac{1}{3}\pi^2\right) \chi_{[0,\frac{1}{2})}, \qquad x_2 = \omega_2 - \left(4 - \frac{1}{3}\pi^2\right) \chi_{[0,\frac{1}{2})} + \omega_{12},$$

$$p_1 = f_1, \qquad \text{and} \qquad p_2 = \chi_{[0,\frac{1}{2})} p_1 + \frac{3}{4} \chi_{[\frac{1}{2},1]}. \tag{1}$$

It is first useful to note that $4 - \pi^2/3 = 2\int_0^1 \omega_{12}\left(s\right) ds$ as confirmed below given that the sum $\sum_{n=1}^{\infty} n^{-2} = \pi^2/6$ is just the Riemann zeta function evaluated at 2 (see, e.g., *Finch*, 2003):

$$\int_0^1 \omega_{12}\left(s\right) ds = \int_0^1 \sum_{n=0}^{\infty} \frac{n}{n+1} \chi_{I_n}\left(s\right) ds$$

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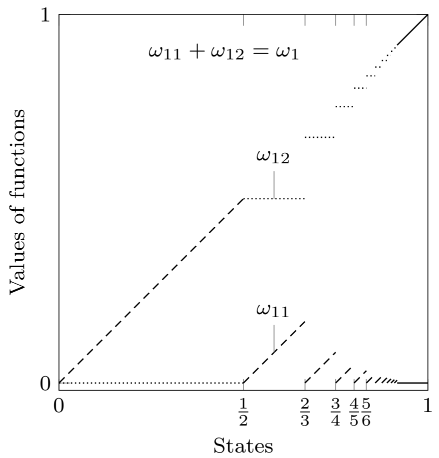

Fig. 1 Decomposition of ω_1

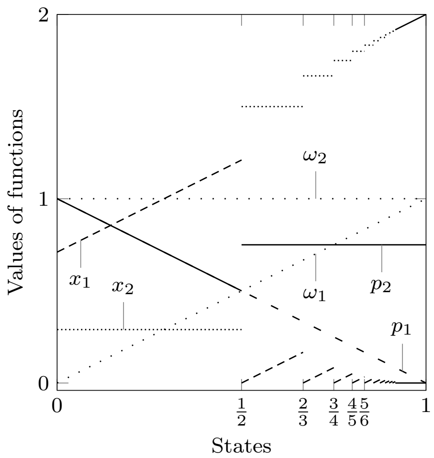

Fig. 2 Personalized equilibrium

$$
\begin{aligned}
&= \sum_{n=0}^{\infty} \int_{I_n} \frac{n}{n+1} ds \\
&= \sum_{n=0}^{\infty} \frac{n}{n+1} \int_{I_n} ds \\
&= \sum_{n=0}^{\infty} \frac{n}{n+1} \left(\frac{n+1}{n+2} - \frac{n}{n+1} \right) \\
&= \sum_{n=0}^{\infty} \left(\frac{(n+1)^2}{(n+2)^2} - \frac{n^2}{(n+1)^2} - \frac{1}{(n+2)^2} \right) \\
&= \lim_{N\to\infty} \left(\sum_{n=0}^{N} \left(\frac{(n+1)^2}{(n+2)^2} - \frac{n^2}{(n+1)^2} \right) - \sum_{n=0}^{N} \frac{1}{(n+2)^2} \right) \\
&= \lim_{N\to\infty} \left(\frac{(N+1)^2}{(N+2)^2} - \sum_{n=0}^{N} \frac{1}{(n+2)^2} \right) \\
&= \lim_{N\to\infty} \left(\frac{(N+1)^2}{(N+2)^2} - \sum_{n=1}^{N} \frac{1}{n^2} + 1 - \frac{1}{(N+1)^2} - \frac{1}{(N+2)^2} \right) \\
&= 2 - \sum_{n=1}^{\infty} \frac{1}{n^2} \\
&= 2 - \frac{1}{6}\pi^2.
\end{aligned}
$$

Clearly, the pair $(x_1, x_2) \in X_1 \times X_2$ is an allocation. To see that condition (a) in the definition of personalized equilibrium is satisfied, notice that

$$\psi_p \cdot x \geq p_i \cdot x > p_i \cdot x_i = \psi_p \cdot x_i. \tag{2}$$

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We verify condition (b) by demonstrating that for all $\alpha \in \mathbb{R}^2_+$ and $y \in X_1 \times X_2$ such that $y_1 + y_2 \le \alpha_1\omega_1 + \alpha_2\omega_2$ we have $p_1 \cdot y_1 + p_2 \cdot y_2 \le \alpha_1 p_1 \cdot x_1 + \alpha_2 p_2 \cdot x_2$:

$$\begin{aligned}
\alpha_1 p_1 \cdot x_1 + \alpha_2 p_2 \cdot x_2 &= \alpha_1 p_1 \cdot \omega_{11} + \alpha_1 p_1 \cdot \left(4 - \frac{1}{3}\pi^2\right) \chi_{[0,\frac{1}{2})} + \alpha_2 p_2 \cdot x_2 \\
&= \alpha_1 p_1 \cdot \omega_{11} + \alpha_1 \frac{3}{8}\left(4 - \frac{1}{3}\pi^2\right) + \alpha_2 p_2 \cdot x_2 \\
&= \alpha_1 p_1 \cdot \omega_{11} + \alpha_1 \frac{3}{4}\int_0^1 \omega_{12}(s)\, ds + \alpha_2 p_2 \cdot x_2 \\
&= \alpha_1 p_1 \cdot \omega_{11} + \alpha_1 p_2 \cdot \omega_{12} + \alpha_2 p_2 \cdot \omega_2 \\
&= p_1 \cdot (\alpha_1\omega_1 - \alpha_1\omega_{12}) + p_2 \cdot (\alpha_2\omega_2 + \alpha_1\omega_{12}) \\
&\ge p_1 \cdot y_1 + p_2 \cdot y_2.
\end{aligned}$$

Finally, condition (c) is also satisfied as $\psi_p \cdot (\omega_1 + \omega_2) \ge p_2 \cdot \omega_2 > 0$.

3.3.2 Equilibrium with Multiple Markets

Let us describe the above personalized equilibrium in terms of two markets as outlined in the last two paragraphs of Subsection 3.2. Below we explain and illustrate in Figure 3 how marketwise prices $p'_1 = p_1$ and $p'_2 = (3/4)\,\chi_{[0,1]}$ clear the two markets. This idea is later developed into a general definition of equilibrium with multiple markets in Section 4.

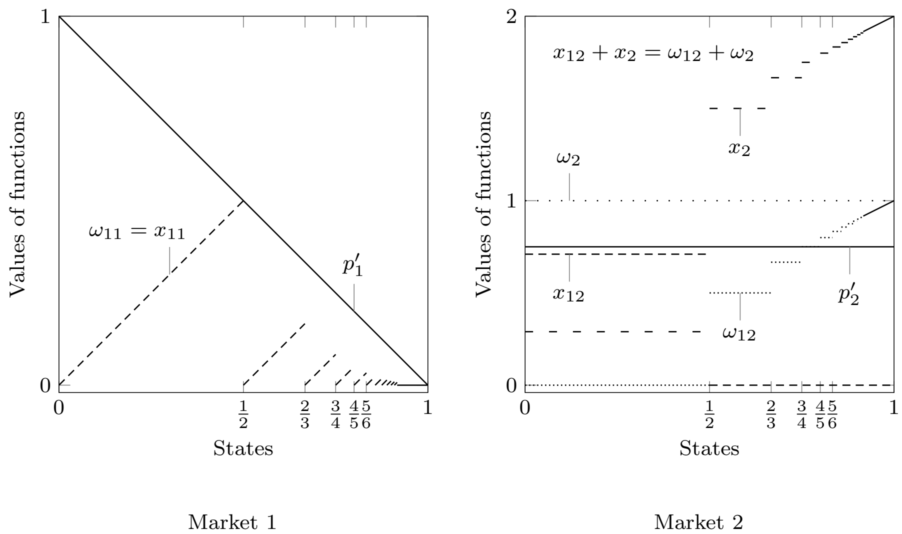

Fig. 3 Equilibrium with multiple markets

First observe that all commodity bundles belonging to X_2 are at least as expensive in market 2 as in market 1. Next notice that $\omega_{12} = \sup(X_2 \cap [0, \omega_1])$.

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Thus we say that selling ω_{12} in market 2 and the remainder ω_{11} in market 1 maximizes revenue of agent 1. The maximum revenue of agent 2 is $p_2' \cdot \omega_2$.

Let agent 1 buy

$$x_{11} = \omega_{11} \text{ in market 1 and } x_{12} = \left(4 - \frac{1}{3}\pi^2\right) \chi_{[0,\frac{1}{2})} \text{ in market 2,}$$

and agent 2 buy x_2 in market 2. Observe that both markets are cleared, i.e. $x_{11} = \omega_{11}$ and $x_{12} + x_2 = \omega_{12} + \omega_2$. Next we demonstrate how both agents' preferences are maximized subject to their budget constraints. These purchases are indeed affordable, as $p_1' \cdot x_{11} + p_2' \cdot x_{12} = p_1' \cdot \omega_{11} + p_2' \cdot \omega_{12}$ and $p_2' \cdot x_2 = p_2' \cdot \omega_2$. Moreover, agent 2 cannot afford commodity bundles $y \in X_2$ such that $u_2(y) > u_2(x_2)$, as $p_2' \cdot y > p_2' \cdot x_2 = p_2' \cdot \omega_2$. To see how agent 1's utility is also maximized, consider any $y_1 \in X_1$ and $y_2 \in X_2$ such that $u_1(y_1 + y_2) > u_1(x_{11} + x_{12})$. We have

$$\begin{aligned} p_1' \cdot y_1 + p_2' \cdot y_2 &\geq p_1' \cdot y_1 + p_1' \cdot y_2 = p_1' \cdot (y_1 + y_2) > p_1' \cdot (x_{11} + x_{12}) \\ &= p_1' \cdot x_{11} + p_1' \cdot x_{12} = p_1' \cdot x_{11} + p_2' \cdot x_{12} = p_1' \cdot \omega_{11} + p_2' \cdot \omega_{12}, \end{aligned}$$

meaning that agent 1 cannot afford a better consumption as well.

3.3.3 Nonexistence of Single Market Linear Equilibrium

We show that no linear price (functional) q on the commodity space and no $(y_1, y_2) \in X_1 \times X_2$ which is an allocation with $y_1 + y_2 = \omega_1 + \omega_2$ are such that

(a) $q \neq 0$,
(b) $y_2 \neq 0$, and
(c) for all i and $y \in X_i$, we have $u_i(y) > u_i(y_i) \implies q \cdot y > q \cdot y_i$.

Supposing the existence of such a q and (y_1, y_2) leads to the following contradiction. First notice that

$$X_2 \cap [0, \omega_{11}] = \{0\}. \tag{3}$$

It follows that $y_2 \leq \omega_2 + \omega_{12}$, since otherwise we have $0 < y_2 \vee (\omega_2 + \omega_{12}) - (\omega_2 + \omega_{12}) \leq \omega_{11}$, violating (3). Now we see that

$$y_1 = \omega_1 + \omega_2 - y_2 \geq \omega_1 + \omega_2 - (\omega_2 + \omega_{12}) = \omega_{11},$$

implying that $y_1(s) > 0$ almost everywhere. Since q must be positive and thus continuous, we may assume without loss of generality that $q = f_1$. But this is indeed impossible given that $y_2 > 0$.

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4 General Model

In the above example of two markets, an agent either participates in a market fully or is not involved at all. Whereas this market structure admits a linear price equilibrium, the alternative of a single market with partial participation does not in view of Subsection 3.3.3. That is why we cannot eschew embedding this requirement in our general model, which is presented next. In our model, agents' heterogeneous involvement in multiple markets is the reason for the differences in consumption sets. Such situations cannot be modelled as partial participation in a single market because of the nonexistence problem.

4.1 Mathematical Preliminaries

A *Banach lattice* is an ordered Banach space L such that every pair $x, y \in L$ has a supremum and an infimum ($\sup\{x, -x\}$ is written $|x|$), as well as satisfies

$$|x| \leq |y| \implies \|x\| \leq \|y\| .$$

For example, the commodity space $L_1([0,1], \Lambda, \lambda)$ in Section 3 is a Banach lattice. Every pair $x, y \in L$ defines the set $[x, y] = \{z \in L : x \leq z \leq y\}$. Sets of this form are called *order intervals*. A *vector sublattice* of a Banach lattice L is a vector subspace closed under pairwise suprema and infima taken in L. For instance, the space $L_1([0,1], \Lambda_2, \lambda)$ is a closed vector sublattice of the commodity space $L_1([0,1], \Lambda, \lambda)$. A Banach lattice is said to be *order complete* if every nonempty subset that is order bounded from above has a supremum. Order completeness is implied by weak compactness of order intervals (*Aliprantis and Border*, 2006, Theorem 9.22). In our example of a Banach lattice, order intervals are weakly compact.

4.2 Economy with Multiple Markets

A *market* is a pair (Z, J) consisting of a set Z of admissible trades and a set J of participants, e.g. $Z = L_1([0,1], \Lambda_2, \lambda)$ and $J = \{1, 2\}$. We consider an *economy* composed of a finite number of markets $(Z_1, J_1), (Z_2, J_2), \ldots, (Z_M, J_M)$. Participants of all markets form the set $I = \cup_{m=1}^{M} J_m$ of *agents*. The *trade spaces* $Z_1, Z_2, \ldots, Z_M$ are subspaces of a *commodity space* L. Their Cartesian product $\boldsymbol{L} = \prod_{m=1}^{M} Z_m$ is called the *market space*. Technically, we assume that

(a) L is a Banach lattice whose order intervals are weakly compact (see remarks preceding Theorem 2),
(b) each Z_m is a closed vector sublattice of L, and
(c) each J_m is nonempty and finite.

Agent i consumes nonnegative bundles from the markets in which she participates. We index these markets by the set $M_i = \{m \in \{1, 2, \ldots, M\} : i \in J_m\}$. Nonnegative bundles traded in market (Z_m, J_m) comprise the positive cone Z_m^+

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of the trade space Z_m. The agent's *consumption set* is $X_i = \sum_{m \in M_i} Z_m^+$. The agent has a *consumption preference correspondence* $P_i : X_i \twoheadrightarrow X_i$, with $P_i(x)$ interpreted as the set of bundles strictly preferred to x. The *initial endowment* of the agent is a consumption bundle $\omega_i \in X_i \setminus \{0\}$. We let $\omega = \sum_{i \in I} \omega_i$ denote the total initial endowment.

4.3 Equilibrium

Agent i's *demand set* $\boldsymbol{X}_i$ is obtained by replacing the coordinate spaces of $\boldsymbol{L}$ having $i \in J_m$ with Z_m^+ and the remaining coordinate spaces of $\boldsymbol{L}$ with $\{0\} \subset \boldsymbol{L}$. An element $\boldsymbol{x} \in \boldsymbol{X}_i$ will usually represent the agent's purchases in different markets for the sake of consumption, with $\boldsymbol{x}_m$ interpreted as the bundle bought in the m^{th} market. There is a natural *consumption mapping* $c : \boldsymbol{L} \to L$ defined by $c(\boldsymbol{x}) = \sum_{m=1}^{M} \boldsymbol{x}_m$, and c_i denotes the restriction of c to $\boldsymbol{X}_i$. We define the induced *demand preference correspondence* $\boldsymbol{P}_i : \boldsymbol{X}_i \twoheadrightarrow \boldsymbol{X}_i$ by posing $\boldsymbol{P}_i(\boldsymbol{x}) = c_i^{-1}(P_i(c_i(\boldsymbol{x})))$.

Agent i's *supply set* $\boldsymbol{Y}_i = c_i^{-1}([0, \omega_i])$ captures how the agent's initial endowment can be sold in different markets. A coordinate $\boldsymbol{y}_m$ of an element $\boldsymbol{y} \in \boldsymbol{Y}_i$ is interpreted as the bundle sold in the m^{th} market.

The value of trades $\boldsymbol{x} = (\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_M) \in \boldsymbol{L}$ is given by a price system prevailing across markets. Formally, a *price system* is a continuous linear functional $\boldsymbol{p}$ on $\boldsymbol{L}$.

We define $\boldsymbol{X} = \prod_{i \in I} \boldsymbol{X}_i$ and $\boldsymbol{Y} = \prod_{i \in I} \boldsymbol{Y}_i$. An *allocation* is a vector $(\boldsymbol{x}, \boldsymbol{y}) \in \boldsymbol{X} \times \boldsymbol{Y}$ such that $\sum_{i \in I} \boldsymbol{x}_i \leq \sum_{i \in I} \boldsymbol{y}_i$.

A *quasi-equilibrium* is a triple $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}, \bar{\boldsymbol{p}})$ consisting of an allocation $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}})$ and a nonzero price system $\bar{\boldsymbol{p}}$ with the following properties for every agent $i \in I$:

(a) $\boldsymbol{y} \in \boldsymbol{Y}_i$ implies $\bar{\boldsymbol{p}} \cdot \boldsymbol{y} \leq \bar{\boldsymbol{p}} \cdot \bar{\boldsymbol{y}}_i$,
(b) $\bar{\boldsymbol{p}} \cdot \bar{\boldsymbol{x}}_i \leq \bar{\boldsymbol{p}} \cdot \bar{\boldsymbol{y}}_i$, and
(c) $\boldsymbol{x} \in \boldsymbol{P}_i(\bar{\boldsymbol{x}}_i)$ implies $\bar{\boldsymbol{p}} \cdot \boldsymbol{x} \geq \bar{\boldsymbol{p}} \cdot \bar{\boldsymbol{y}}_i$.

A quasi-equilibrium $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}, \bar{\boldsymbol{p}})$ is said to be an *equilibrium* if for every agent $i \in I$ and for all $\boldsymbol{x} \in \boldsymbol{P}_i(\bar{\boldsymbol{x}}_i)$ we have $\bar{\boldsymbol{p}} \cdot \boldsymbol{x} > \bar{\boldsymbol{p}} \cdot \bar{\boldsymbol{y}}_i$. A quasi-equilibrium $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}, \bar{\boldsymbol{p}})$ is said to be *nontrivial* if there exist an agent $i \in I$ and an $\boldsymbol{x} \in \boldsymbol{X}_i$ such that $\bar{\boldsymbol{p}} \cdot \boldsymbol{x} < \bar{\boldsymbol{p}} \cdot \bar{\boldsymbol{x}}_i$.

5 General Perspective on Reinterpretation

Now we are ready for a general analysis of the ideas that in Section 3 allow us to reinterpret an example of nonlinear pricing in personalized equilibrium. Theorem 1 generalizes such reinterpretation under economically meaningful conditions, and we relate one of them to the role of economic intermediation.

We start by introducing nonlinear pricing independently of our general model of multiple markets (Section 4), and only keep the same *commodity*

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space L. *Agents* now collectively constitute a finite nonempty set $\hat{I}$. Their *consumption sets* $\hat{X}_i \subset L$, one for each agent $i \in \hat{I}$, enter this model directly as its primitives, in contrast to our more structural model of multiple markets. The *preference correspondence* $\hat{P}_i : \hat{X}_i \twoheadrightarrow \hat{X}_i$ of every agent $i \in \hat{I}$ is analogous to our consumption preference correspondences. Also analogously, the *initial endowment* of the agent is a consumption bundle $\hat{\omega}_i \in \hat{X}_i$. We define $\hat{X} = \prod_{i\in\hat{I}} \hat{X}_i$, let $\hat{\omega} = \sum_{i\in\hat{I}} \hat{\omega}_i$ stand for the total initial endowment, and associate with every commodity bundle $x \in L$ the set

$$\mathcal{A}(x) = \left\{ y \in \hat{X} : \sum_{i\in\hat{I}} y_i \leq x \right\}.$$

Technically, we assume for every agent $i \in \hat{I}$ that

(a) $\hat{X}_i$ is the positive cone of a closed vector sublattice $\hat{Z}_i$ of L, and
(b) $\hat{\omega}_i > 0$.

For our purpose it is convenient to suppose from the outset that a personalized equilibrium exists and then to analyze it. In this equilibrium personalized prices are also continuous linear functionals on the commodity space, i.e. they are elements of the topological dual L' of L. Let $\hat{p} \in (L')^{\hat{I}}$ be the vector of these equilibrium *personalized prices*. They induce a possibly *nonlinear price* $\psi_{\hat{p}} : \sum_{i\in\hat{I}} \hat{X}_i \to \mathbb{R}_+$ by assigning to a commodity bundle x in the domain the value

$$\psi_{\hat{p}} \cdot x = \sup \left\{ \sum_{i\in\hat{I}} \hat{p}_i \cdot y_i : y \in \mathcal{A}(x) \right\}.$$

Simultaneously, agents' choices determine the equilibrium allocation $\hat{x} \in \mathcal{A}(\hat{\omega})$ such that the following conditions of *Aliprantis et al.* (2001) hold:

(a) for all $i \in \hat{I}$, if $x \in \hat{P}_i(\hat{x}_i)$, then $\psi_{\hat{p}} \cdot x > \psi_{\hat{p}} \cdot \hat{x}_i$,
(b) for all $\alpha \in \mathbb{R}_+^{\hat{I}}$, we have $\psi_{\hat{p}} \cdot \sum_{i\in\hat{I}} \alpha_i \hat{\omega}_i \leq \sum_{i\in\hat{I}} \alpha_i \psi_{\hat{p}} \cdot \hat{x}_i$, and
(c) $\psi_{\hat{p}} \cdot \hat{\omega} > 0$.

Let us refer to this *personalized equilibrium* by the pair $(\hat{x}, \hat{p})$.

We say that the personalized equilibrium is *segmentable* if it can be modelled as an equilibrium $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}, \bar{\boldsymbol{p}})$ of an economy with multiple markets in the sense that

(a) $I = \hat{I}$,
(b) for every agent $i \in I$, we have $X_i = \hat{X}_i$, $P_i = \hat{P}_i$, $\omega_i = \hat{\omega}_i$, as well as $c(\bar{\boldsymbol{x}}_i) = \hat{x}_i$, and
(c) for every commodity bundle x in the domain of the price $\psi_{\hat{p}}$, we have

$$\psi_{\hat{p}} \cdot x = \sup \left\{ \bar{\boldsymbol{p}} \cdot \boldsymbol{x} : \boldsymbol{x} \in c^{-1}([0, x]), \boldsymbol{x} \geq 0 \right\}. \tag{4}$$

Such segmentability is established in Theorem 1 under two conditions, which we next introduce and motivate.

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Individual supportability. Looking carefully into the proofs of *Aliprantis et al.* (2001), one finds that their conclusions are stronger than those stated as theorems in the following sense. Their assumptions yield the existence of personalized equilibrium with personalized prices summarizing individual incomes and substitution attitudes (p. 44) via

$$\hat{p}_i \cdot \hat{x}_i = \psi_{\hat{p}} \cdot \hat{x}_i, \text{ and } x \in \hat{P}_i(\hat{x}_i) \implies \hat{p}_i \cdot x > \hat{p}_i \cdot \hat{x}_i. \tag{5}$$

This way, personalized prices are closer to Walrasian, as in the example of Section 3 in view of (2). We say that the personalized equilibrium $(\hat{x}, \hat{p})$ is *individually supporting* if conditions (5) hold for every agent $i \in \hat{I}$. This condition may be violated, but individually supporting personalized equilibria exist under the assumptions of *Aliprantis et al.* (2001). That is why we explicitly assume individual supportability.

Bilateral feasibility. To motivate this concept, consider first the benchmark case in which all consumption sets coincide with the positive cone of L and there is no disposal, i.e.

$$\sum_{i \in \hat{I}} \hat{x}_i = \hat{\omega}.$$

Here $\hat{x}$ is bilaterally feasible in the sense that some $z \in \hat{X}^{\hat{I}}$ decomposes individually

$$\hat{\omega}_i = \sum_{j \in \hat{I}} z_{ij} \text{ and } \hat{x}_i = \sum_{j \in \hat{I}} z_{ji} \tag{6}$$

by the Riesz decomposition property (*Aliprantis and Tourky*, 2007, Theorem 1.54). A coordinate z_{ij} is interpreted as a consumption bundle sold by agent i to agent j. In the example of Section 3, the consumption set of agent 2 differs from the positive cone but there exist $z_{12} \in X_1 \cap X_2$, $z_{21} \in X_2 \cap X_1$, $z_{11} \in X_1$, and $z_{22} \in X_2$ with

$$\begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = \begin{pmatrix} z_{11} + z_{12} \\ z_{21} + z_{22} \end{pmatrix} \text{ and } \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} z_{11} + z_{21} \\ z_{12} + z_{22} \end{pmatrix}.$$

While in general having $z_{ij} \in X_i \cap X_j$ only states that both the buyer and the seller can consume this traded bundle, it says more with differential information. In this context, the condition ensures that both agents are sufficiently informed to verify ex-post consequences of the ex-ante agreement z_{ij}, due to measurability. The following definition generalizes this property. We say that the personalized equilibrium $(\hat{x}, \hat{p})$ is *bilaterally feasible* if there exist vectors $z_{ij} \in \hat{X}_i \cap \hat{X}_j$, one for each pair $(i, j) \in \hat{I}^2$, satisfying (6) for all $i \in \hat{I}$.

Lest the indispensability of bilateral feasibility be a concern, a comforting observation is that a natural way forward suggests itself in that case. Bilaterally infeasible allocations require intermediation, and modelling intermediation

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explicitly may prove fruitful. In particular, it may be helpful for explaining the existence of intermediaries in the real world.

The segmentation obtained in the proof of Theorem 1 is *maximal* in the sense that each pair of agents forms a separate market, as defined in (7). Even though equilibrium prices may differ across pairs, in some sense they are consistent with perfect competition if there are many pairs. This is because agents cannot do better by unilaterally changing their supply and demand across pairs given those prices, by our definition of equilibrium.

Simpler representations of the personalized equilibrium are possible if consumption sets do not differ too much. One way is to combine markets having identical trade spaces Z_m in the maximal segmentation, letting their participants (the union of the corresponding sets J_m) redefine the set of participants. In such a refinement, prices would form in a number of coalitions, and we note a related model of *Dagan et al.* (2000) where agents bargain in coalitions.

Theorem 1 *The personalized equilibrium* $(\hat{x}, \hat{p})$ *is segmentable if it is individually supporting and bilaterally feasible.*

Proof We let markets be given by an arbitrary enumeration of the family

$$\left\{ \left(\hat{Z}_i \cap \hat{Z}_j, \{i, j\} \right) : i, j \in \hat{I} \right\}. \tag{7}$$

An allocation $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}})$ such that

$$\sum_{i \in I} \bar{\boldsymbol{x}}_i = \sum_{i \in I} \bar{\boldsymbol{y}}_i \tag{8}$$

is defined by applying to coordinates having $J_m = \{i, j\}$ the formulas $\bar{\boldsymbol{x}}_{im} = z_{ji}$ and $\bar{\boldsymbol{y}}_{im} = z_{ij}$. A price system $\bar{\boldsymbol{p}}$ is defined by considering the topological dual of each Z_m as a lattice, in which we take the supremum $\bar{\boldsymbol{p}}_m = \bigvee_{i \in J_m} (\hat{p}_i|_{Z_m})$, and letting

$$\bar{\boldsymbol{p}} \cdot \boldsymbol{x} = \sum_{m=1}^{M} \bar{\boldsymbol{p}}_m \cdot \boldsymbol{x}_m. \tag{9}$$

If $\boldsymbol{x} \geq 0$, then we have

$$\begin{aligned} \bar{\boldsymbol{p}} \cdot \boldsymbol{x} &= \sum_{m=1}^{M} \bar{\boldsymbol{p}}_m \cdot \boldsymbol{x}_m = \sum_{m=1}^{M} \left(\bigvee_{i \in J_m} (\hat{p}_i|_{Z_m}) \right) \cdot \boldsymbol{x}_m \\ &\leq \sum_{m=1}^{M} \psi_{\hat{p}} \cdot \boldsymbol{x}_m \leq \psi_{\hat{p}} \cdot \sum_{m=1}^{M} \boldsymbol{x}_m = \psi_{\hat{p}} \cdot c(\boldsymbol{x}). \end{aligned} \tag{10}$$

By this fact, the definition of supply set, condition (b) in the definition of personalized equilibrium, and individual supportability, all i and $\boldsymbol{y} \in \boldsymbol{Y}_i$ satisfy

$$\bar{\boldsymbol{p}} \cdot \boldsymbol{y} \leq \psi_{\hat{p}} \cdot c(\boldsymbol{y}) \leq \psi_{\hat{p}} \cdot \hat{\omega}_i \leq \psi_{\hat{p}} \cdot \hat{x}_i = \hat{p}_i \cdot \hat{x}_i. \tag{11}$$

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For all i and $\boldsymbol{x} \in \boldsymbol{X}_i$, we additionally obtain

$$\bar{\boldsymbol{p}} \cdot \boldsymbol{x} = \sum_{m=1}^{M} \bar{\boldsymbol{p}}_m \cdot \boldsymbol{x}_m \geq \sum_{m=1}^{M} \hat{p}_i \cdot \boldsymbol{x}_m = \hat{p}_i \cdot \sum_{m=1}^{M} \boldsymbol{x}_m = \hat{p}_i \cdot c\,(\boldsymbol{x})\,, \qquad (12)$$

and next verify conditions (a)–(c) in the definition of quasi-equilibrium.

(c) Combining (12), individual supportability, and (11) for $\boldsymbol{y} = \bar{\boldsymbol{y}}_i$ yields

$$\bar{\boldsymbol{p}} \cdot \boldsymbol{x} \geq \hat{p}_i \cdot c\,(\boldsymbol{x}) > \hat{p}_i \cdot \hat{x}_i \geq \bar{\boldsymbol{p}} \cdot \bar{\boldsymbol{y}}_i, \qquad (13)$$

as required.

(b) Results (11) and (12) for $\boldsymbol{y} = \bar{\boldsymbol{y}}_i$ and $\boldsymbol{x} = \bar{\boldsymbol{x}}_i$ give us $\bar{\boldsymbol{p}} \cdot \bar{\boldsymbol{x}}_i \geq \bar{\boldsymbol{p}} \cdot \bar{\boldsymbol{y}}_i$, while an application of equation (8) yields $\sum_{i \in I} (\bar{\boldsymbol{p}} \cdot \bar{\boldsymbol{x}}_i - \bar{\boldsymbol{p}} \cdot \bar{\boldsymbol{y}}_i) = 0$. It follows for each i that $\bar{\boldsymbol{p}} \cdot \bar{\boldsymbol{x}}_i = \bar{\boldsymbol{p}} \cdot \bar{\boldsymbol{y}}_i$, establishing the equilibrium condition.

(a) Starting with the last condition, letting $\boldsymbol{x} = \bar{\boldsymbol{x}}_i$ in (12), and invoking (11), we obtain

$$\bar{\boldsymbol{p}} \cdot \bar{\boldsymbol{y}}_i \geq \bar{\boldsymbol{p}} \cdot \bar{\boldsymbol{x}}_i \geq \hat{p}_i \cdot c\,(\bar{\boldsymbol{x}}_i) = \hat{p}_i \cdot \hat{x}_i \geq \bar{\boldsymbol{p}} \cdot \boldsymbol{y},$$

as desired.

Due to the strict inequality in (13), the quasi-equilibrium $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}, \bar{\boldsymbol{p}})$ is actually an equilibrium. Now it only remains to verify formula (4). Firstly, every $y \in \mathcal{A}\,(x)$ defines an $\boldsymbol{x} \in c^{-1}\,([0,x])$ by matching coordinates so that $\boldsymbol{x}_m = y_i \geq 0$ if $J_m = \{i\}$ and $\boldsymbol{x}_m = 0$ otherwise. Since $\sum_{i \in I} \hat{p}_i \cdot y_i = \sum_{m=1}^{M} \bar{\boldsymbol{p}}_m \cdot \boldsymbol{x}_m = \bar{\boldsymbol{p}} \cdot \boldsymbol{x}$, the supremum on the right hand side of equation (4) is greater than or equal to $\psi_{\hat{p}} \cdot x$. The reverse inequality is obtained using (10) and calculating for every $\boldsymbol{x} \in c^{-1}\,([0,x])$ satisfying $\boldsymbol{x} \geq 0$ that $\bar{\boldsymbol{p}} \cdot \boldsymbol{x} \leq \psi_{\hat{p}} \cdot c\,(\boldsymbol{x}) \leq \psi_{\hat{p}} \cdot x$. This yields (4) and completes the proof. □

6 General and Direct Equilibrium Existence Result

A corollary of Theorem 1 is equilibrium existence for economies with multiple markets fitted in a particular fashion to bilaterally feasible personalized equilibria. But operation (7) and its refinements allow modelling all other economies from the nonlinear equilibrium theory in focus of Section 5 in terms of multiple markets. Thus our general existence result without feasibility restrictions in Theorem 2 establishes our model as a nonvacuous natural alternative to nonlinear pricing. In our approach, one can assume that each market has a "government institution" able to enforce contracts, which is problematic in the single market model (*Podczeck and Yannelis*, 2008). Besides, our framework accommodates economic situations with evident structures of multiple markets, e.g. the world economy with barriers to international trade. Closed economy examples are furnished by market exclusions via eligibility criteria in financial services, license or qualification requirements, or age restrictions.

One could be inclined to misjudge Theorem 2 by viewing each supply set $\boldsymbol{Y}_i$ as a production set of a firm owned privately by agent i. Indeed, our exchange

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economy with multiple markets may be viewed as a production economy in $\boldsymbol{L}$. Unfortunately, there is no result on the existence of production equilibria that we could apply directly. The reason is that consumption sets in our induced production economy may be thin in the sense that they need not coincide with the positive cone, which also may have an empty interior. Even though consumption sets are allowed to be thin in *Florenzano and Marakulin* (2001) and *Aliprantis et al.* (2006), their properness assumptions are too strong when consumption sets are actually thin. Nevertheless, it is convenient to keep this analogy with production economies in mind when proving the existence of equilibrium with multiple markets.

One technical assumption on the commodity space in the general model of Section 4 is weak compactness of order intervals. Even single market equilibrium existence theorems require this property partially but indispensably, as *Aliprantis et al.* (1987) show in their Example 5.7. They describe an economy which lacks this property but otherwise fits our framework with a single market and satisfies all our remaining existence conditions (Theorem 2). Simply missing weak compactness, this economy does not have an equilibrium, as the results of *Aliprantis et al.* (1987) let us see. When preferences have utility representations, a common alternative assumption is compactness of the individually rational utility set, used recently by *Xanthos* (2014).

The additional assumptions required by Theorem 2, which is stated below, are standard. For instance, Assumptions (a)–(d) are used by *Aliprantis et al.* (2001). Assumption (e) is satisfied if P_i has open values, which is also assumed by *Aliprantis et al.* (2001). Assumption (f) is a version of the properness condition introduced by *Tourky* (1998). We also remark that a nontrivial quasi-equilibrium is an equilibrium if irreducibility assumptions are satisfied (see *Florenzano*, 2003, Section 5.3.6).

Theorem 2 *There exists a nontrivial quasi-equilibrium if the following conditions hold for every agent $i \in I$:*

(a) P_i is irreflexive, i.e. $x \notin P_i(x)$ for all $x \in X_i$;
(b) P_i is convex-valued, i.e. $P_i(x)$ is a convex set for all $x \in X_i$;
(c) P_i is monotone, i.e. $x + z \in P_i(x)$ for all $x \in X_i$ and $z \in X_i \setminus \{0\}$;
(d) P_i has weakly open lower sections, i.e. $\{z \in X_i : x \in P_i(z)\}$ is weakly open in X_i for all $x \in X_i$;
(e) P_i is "continuous" in the sense that $x \in X_i$, $z \in P_i(x)$, and $z' \in X_i$ implies $\alpha z + (1-\alpha) z' \in P_i(x)$ for some scalar $\alpha \in [0,1)$;
(f) P_i is proper in the sense that there exists a convex-valued correspondence $\hat{P}_i : X_i \twoheadrightarrow L$ such that for all $x \in X_i$
  (i) $x + \omega_i$ is an interior point of $\hat{P}_i(x)$ and
  (ii) $\hat{P}_i(x) \cap X_i = P_i(x)$.

Proof of Theorem 2

Lemma 1 *For every agent $i \in I$, the supply set $\boldsymbol{Y}_i$ has a supremum $\boldsymbol{u}_i$ in $\boldsymbol{L}$.*

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Proof Due to order completeness of L, the set $\boldsymbol{Y}_i$ has a supremum $\boldsymbol{u}_i \in L^M$. We prove the lemma by showing that $\boldsymbol{u}_i \in \boldsymbol{L}$.

For every m, let $Y_m = Z_m \cap [0, \omega_i]$. Notice that $\boldsymbol{u}_{im} = \sup Y_m \in L$ if $i \in J_m$ and $\boldsymbol{u}_{im} = 0$ otherwise. It suffices to show that $\boldsymbol{u}_{im} \in Z_m$ if $i \in J_m$. Observe that Y_m is directed by the order relation $\geq$ of L, making the identity function on Y_m a net in $[0, \omega_i]$. Since this order interval is weakly compact, the net has a subnet $\{y_\alpha\}_{\alpha \in D}$ which converges weakly to some $y \in [0, \omega_i]$. Since Z_m is closed and convex, it is weakly closed as well, and it follows that $y \in Z_m \cap [0, \omega_i]$. This also means that $y \leq \boldsymbol{u}_{im}$, and we complete the proof by demonstrating the reverse inequality $y \geq \boldsymbol{u}_{im}$. Let us denote the direction on D by $\succeq$. Consider another binary relation $\gtrsim$ on D defined by posing $\alpha \gtrsim \beta$ if and only if $\alpha \succeq \beta$ and $y_\alpha \geq y_\beta$. This binary relation $\gtrsim$ inherits reflexivity and transitivity from $\succeq$ and $\geq$. Thus $\gtrsim$ is also a direction provided that every pair $\alpha, \beta \in D$ has an upper bound in $(D, \gtrsim)$. Such an upper bound γ can be constructed as follows: let $z = y_\alpha \vee y_\beta \in Y_m$, using the definition of subnet take any $\delta_0 \in D$ such that $\delta \succeq \delta_0$ implies $y_\delta \geq z$, and let γ be an upper bound of $\{\alpha, \beta, \delta_0\}$ in $(D, \succeq)$. Now consider the net $\{y_\alpha\}_{\alpha \in (D, \gtrsim)}$, which also converges weakly to y. Moreover, this net is increasing, i.e. $\alpha \gtrsim \beta$ implies $y_\alpha \geq y_\beta$. Our last two observations reveal that $y = \sup \{y_\alpha : \alpha \in D\}$ (*Aliprantis and Tourky*, 2007, part (4) of Lemma 2.3). Now the definition of subnet yields for each $z \in Y_m$ some $\alpha \in D$ such that $y \geq y_\alpha \geq z$. This confirms that $y \geq \boldsymbol{u}_{im}$. □

For each i, let $\boldsymbol{u}_i \geq 0$ be the supremum given by Lemma 1. We define $\boldsymbol{u} = \sum_{i \in I} \boldsymbol{u}_i \in \boldsymbol{L}$, take the order interval $[-\boldsymbol{u}, \boldsymbol{u}]$ in $\boldsymbol{L}$, let $\boldsymbol{K} = \cup_{n=1}^{\infty} n [-\boldsymbol{u}, \boldsymbol{u}]$ be the principal ideal generated by $\boldsymbol{u}$ in $\boldsymbol{L}$, and $\boldsymbol{K}_+ = \{\boldsymbol{x} \in \boldsymbol{K} : \boldsymbol{x} \geq 0\}$.

At this stage, let us view $\boldsymbol{K}$ as the commodity space of the production economy which we now construct by restricting our economy with multiple markets. Agents in the production economy are the same as in the economy with multiple markets. For each i, the restricted demand set $\boldsymbol{X}_i \cap \boldsymbol{K}$ is viewed as the agent's consumption set, which is the domain of the agent's preference correspondence $\boldsymbol{x} \mapsto \boldsymbol{P}_i(\boldsymbol{x}) \cap \boldsymbol{K}$. The agent's initial endowment is $0 \in \boldsymbol{X}_i \cap \boldsymbol{K}$ but the agent possesses a private firm whose production set is $\boldsymbol{Y}_i \subset \boldsymbol{K}$. We let the price space be the topological dual of $(\boldsymbol{K}, \|\cdot\|_{\boldsymbol{u}})$, where the norm $\|\cdot\|_{\boldsymbol{u}}$ on $\boldsymbol{K}$ is defined by

$$\|\boldsymbol{x}\|_{\boldsymbol{u}} = \inf \{\alpha \in \mathbb{R}_{++} : \boldsymbol{x} \in \alpha [-\boldsymbol{u}, \boldsymbol{u}]\}$$

(making $\boldsymbol{u}$ an interior point of the positive cone $\boldsymbol{K}_+$). Conveniently, this production economy is a special case of the model in Chapter 5 of *Florenzano* (2003). We combine Propositions 5.2.3 and 5.3.1 there, whose assumptions are easy to verify noting that each i satisfies

$$\emptyset \neq c_i^{-1}(\omega_i) \subset (\boldsymbol{X}_i \cap \boldsymbol{K}) \setminus \{0\},$$

and first obtain a nontrivial quasi-equilibrium in $\boldsymbol{K}$. Namely, we are able to find an allocation $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}) \in \boldsymbol{K}^I \times \boldsymbol{K}^I$ and a positive linear functional $\bar{\boldsymbol{q}}$ on $\boldsymbol{K}$ with

$$\bar{\boldsymbol{q}} \cdot \bar{\boldsymbol{x}}_i > 0 \text{ for some } i \tag{14}$$

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as well as the following properties for each i:

(a) $\boldsymbol{x} \in \boldsymbol{P}_i(\bar{\boldsymbol{x}}_i) \cap \boldsymbol{K}$ implies $\bar{\boldsymbol{q}} \cdot \boldsymbol{x} \geq \bar{\boldsymbol{q}} \cdot \bar{\boldsymbol{y}}_i$,
(b) $\boldsymbol{y} \in \boldsymbol{Y}_i$ implies $\bar{\boldsymbol{q}} \cdot \boldsymbol{y} \leq \bar{\boldsymbol{q}} \cdot \bar{\boldsymbol{y}}_i$, and
(c) $\bar{\boldsymbol{q}} \cdot \bar{\boldsymbol{x}}_i = \bar{\boldsymbol{q}} \cdot \bar{\boldsymbol{y}}_i$.

For each i, define the convex set $\hat{\boldsymbol{P}}_i(\bar{\boldsymbol{x}}_i) = c^{-1}\left(\hat{P}_i(c(\bar{\boldsymbol{x}}_i))\right)$. These sets inherit the following two properties from properness:

(i) elements of $\bar{\boldsymbol{x}}_i + c_i^{-1}(\omega_i)$ belong to the interior of $\hat{\boldsymbol{P}}_i(\bar{\boldsymbol{x}}_i)$, and
(ii) $\hat{\boldsymbol{P}}_i(\bar{\boldsymbol{x}}_i) \cap \boldsymbol{X}_i = \boldsymbol{P}_i(\bar{\boldsymbol{x}}_i)$.

We note that the interior of $\hat{\boldsymbol{P}}_i(\bar{\boldsymbol{x}}_i)$ is convex and define an open convex set

$$\boldsymbol{V}_i = \left\{\alpha(\boldsymbol{z} - \bar{\boldsymbol{x}}_i) : \alpha \in \mathbb{R}_{++}, \boldsymbol{z} \in \operatorname{int}\left(\hat{\boldsymbol{P}}_i(\bar{\boldsymbol{x}}_i)\right)\right\} \subset \boldsymbol{L}.$$

Lemma 2 *For every agent $i \in I$, if we have $\boldsymbol{x} \in (\bar{\boldsymbol{x}}_i + \boldsymbol{V}_i) \cap \boldsymbol{X}_i \cap \boldsymbol{K}$, then $\bar{\boldsymbol{q}} \cdot \boldsymbol{x} \geq \bar{\boldsymbol{q}} \cdot \bar{\boldsymbol{x}}_i$.*

Proof Write $\boldsymbol{x} = \bar{\boldsymbol{x}}_i + \alpha(\boldsymbol{z} - \bar{\boldsymbol{x}}_i)$ for some $\boldsymbol{z} \in \operatorname{int}\left(\hat{\boldsymbol{P}}_i(\bar{\boldsymbol{x}}_i)\right) \cap \boldsymbol{K}$ and $\alpha \in \mathbb{R}_{++}$. Since $\bar{\boldsymbol{x}}_i$ belongs to the closure of $\hat{\boldsymbol{P}}_i(\bar{\boldsymbol{x}}_i)$, it follows that $\bar{\boldsymbol{x}}_i + \beta(\boldsymbol{z} - \bar{\boldsymbol{x}}_i) \in \operatorname{int}\left(\hat{\boldsymbol{P}}_i(\bar{\boldsymbol{x}}_i)\right)$ for all scalars $\beta \in (0, 1]$. On the other hand, taking positive parts on both sides of $\bar{\boldsymbol{x}}_i \geq -\alpha(\boldsymbol{z} - \bar{\boldsymbol{x}}_i)$ shows that $\bar{\boldsymbol{x}}_i \geq \alpha(\boldsymbol{z} - \bar{\boldsymbol{x}}_i)^-$. Let $\beta = \min\{1, \alpha\}$ and observe that

$$\begin{aligned} 0 &\leq \bar{\boldsymbol{x}}_i - \alpha(\boldsymbol{z} - \bar{\boldsymbol{x}}_i)^- \leq \bar{\boldsymbol{x}}_i - \beta(\boldsymbol{z} - \bar{\boldsymbol{x}}_i)^- \\ &\leq \bar{\boldsymbol{x}}_i + \beta(\boldsymbol{z} - \bar{\boldsymbol{x}}_i) \in \operatorname{int}\left(\hat{\boldsymbol{P}}_i(\bar{\boldsymbol{x}}_i)\right) \cap \boldsymbol{X}_i \cap \boldsymbol{K} \subset \boldsymbol{P}_i(\bar{\boldsymbol{x}}_i) \cap \boldsymbol{K}. \end{aligned}$$

Now the restricted quasi-equilibrium properties of $\bar{\boldsymbol{x}}_i$ and $\bar{\boldsymbol{q}}$ yield $\bar{\boldsymbol{q}} \cdot (\boldsymbol{z} - \bar{\boldsymbol{x}}_i) \geq 0$. It follows that $\bar{\boldsymbol{q}} \cdot \boldsymbol{x} \geq \bar{\boldsymbol{q}} \cdot \bar{\boldsymbol{x}}_i$ indeed. □

Lemma 3 *For every agent $i \in I$, there exist a price system $\boldsymbol{p}_i$ and a linear functional $\boldsymbol{p}'_i$ on $\boldsymbol{L}$ such that*

(a) $\boldsymbol{x} \in \bar{\boldsymbol{x}}_i + \boldsymbol{V}_i$ implies $\boldsymbol{p}_i \cdot \boldsymbol{x} \geq \boldsymbol{p}_i \cdot \bar{\boldsymbol{x}}_i$,
(b) $\boldsymbol{x} \in \boldsymbol{X}_i \cap \boldsymbol{K}$ implies $\boldsymbol{p}'_i \cdot \boldsymbol{x} \geq \boldsymbol{p}'_i \cdot \bar{\boldsymbol{x}}_i$, and
(c) $\boldsymbol{p}_i \cdot \boldsymbol{x} + \boldsymbol{p}'_i \cdot \boldsymbol{x} = \bar{\boldsymbol{q}} \cdot \boldsymbol{x}$ for all $\boldsymbol{x} \in \boldsymbol{K}$.

Proof This result follows from Lemma 2 and the Podczeck's extension lemma (*Podczeck*, 1996, Lemma 2). □

For every i, let $\boldsymbol{p}_i$ be a price system given by Lemma 3. For each m, this $\boldsymbol{p}_i$ induces a continuous linear functional $\boldsymbol{p}_{im}$ on Z_m by matching values so that $\boldsymbol{p}_{im} \cdot x = \boldsymbol{p}_i \cdot \boldsymbol{x}$ when $\boldsymbol{x}_m = x$ and $\boldsymbol{x}_n = 0$ for $n \neq m$. Since the topological dual of Z_m is a lattice, the supremum

$$\bar{\boldsymbol{p}}_m = \bigvee_{i \in J_m} \boldsymbol{p}_{im}$$

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is also a continuous linear functional on Z_m. A price system $\bar{\boldsymbol{p}}$ is now defined by formula (9). For all i and $\boldsymbol{x} \in \boldsymbol{X}_i$, we have

$$\boldsymbol{p}_i \cdot \boldsymbol{x} \le \bar{\boldsymbol{p}} \cdot \boldsymbol{x} = \sum_{m=1}^{M} \bar{\boldsymbol{p}}_m \cdot \boldsymbol{x}_m. \tag{15}$$

Lemma 4 *The following statements are true for every agent $i \in I$:*

(a) $\boldsymbol{x} \in \boldsymbol{X}_i \cap \boldsymbol{K}$ implies $\bar{\boldsymbol{q}} \cdot \boldsymbol{x} \ge \boldsymbol{p}_i \cdot \boldsymbol{x}$, and
(b) $\bar{\boldsymbol{q}} \cdot (\bar{\boldsymbol{x}}_i - \boldsymbol{x}) \le \boldsymbol{p}_i \cdot (\bar{\boldsymbol{x}}_i - \boldsymbol{x})$ for all $\boldsymbol{x} \in \boldsymbol{X}_i$ satisfying $\bar{\boldsymbol{x}}_i \ge \boldsymbol{x}$.

Proof (a) Since $\bar{\boldsymbol{x}}_i + \bar{\boldsymbol{x}}_i \in \boldsymbol{X}_i \cap \boldsymbol{K}$, part (b) of Lemma 3 yields $\boldsymbol{p}'_i \cdot \bar{\boldsymbol{x}}_i + \boldsymbol{p}'_i \cdot \bar{\boldsymbol{x}}_i \ge \boldsymbol{p}'_i \cdot \bar{\boldsymbol{x}}_i$, which implies that $\boldsymbol{p}'_i \cdot \bar{\boldsymbol{x}}_i \ge 0$. Applying part (b) of Lemma 3 once again, we see that $\boldsymbol{p}'_i \cdot \boldsymbol{x} \ge \boldsymbol{p}'_i \cdot \bar{\boldsymbol{x}}_i \ge 0$ for all $\boldsymbol{x} \in \boldsymbol{X}_i \cap \boldsymbol{K}$. Combining this result with part (c) of Lemma 3, we obtain the desired conclusion.

(b) Since $\boldsymbol{x} \in \boldsymbol{K}$, part (b) of Lemma 3 yields $\boldsymbol{p}'_i \cdot \boldsymbol{x} \ge \boldsymbol{p}'_i \cdot \bar{\boldsymbol{x}}_i$. Now part (c) of Lemma 3 shows that $\boldsymbol{p}_i \cdot (\bar{\boldsymbol{x}}_i - \boldsymbol{x}) - \bar{\boldsymbol{q}} \cdot (\bar{\boldsymbol{x}}_i - \boldsymbol{x}) = \boldsymbol{p}'_i \cdot (\boldsymbol{x} - \bar{\boldsymbol{x}}_i) \ge 0$, completing the proof. □

Lemma 5 *For all $\boldsymbol{x} \in \boldsymbol{K}_+$, we have $\bar{\boldsymbol{q}} \cdot \boldsymbol{x} \ge \bar{\boldsymbol{p}} \cdot \boldsymbol{x}$.*

Proof For each m, we write $\bar{\boldsymbol{q}}_m \cdot \boldsymbol{x}_m$ to denote the value of $\bar{\boldsymbol{q}}$ at the point $\boldsymbol{y} \in \boldsymbol{K}$ with $\boldsymbol{y}_m = \boldsymbol{x}_m$ and $\boldsymbol{y}_n = 0$ for $n \ne m$. Since $\boldsymbol{x}_m \in Z_m^+$, the Riesz-Kantorovich formula yields

$$\begin{aligned}\bar{\boldsymbol{p}}_m \cdot \boldsymbol{x}_m &= \left(\bigvee_{i \in J_m} \boldsymbol{p}_{im} \right) \cdot \boldsymbol{x}_m \\ &= \sup \left\{ \sum_{i \in J_m} \boldsymbol{p}_{im} \cdot z_i : z \in \left(Z_m^+\right)^{J_m}, \sum_{i \in J_m} z_i = \boldsymbol{x}_m \right\},\end{aligned} \tag{16}$$

and any z in this supremum has a corresponding $\boldsymbol{z} \in \prod_{i \in J_m} (\boldsymbol{X}_i \cap \boldsymbol{K})$ with $\boldsymbol{p}_i \cdot \boldsymbol{z}_i = \boldsymbol{p}_{im} \cdot z_i$ for all i and $\sum_{i \in J_m} \boldsymbol{z}_i = \boldsymbol{y}$. Using part (a) of Lemma 4, we obtain

$$\sum_{i \in J_m} \boldsymbol{p}_{im} \cdot z_i = \sum_{i \in J_m} \boldsymbol{p}_i \cdot \boldsymbol{z}_i \le \sum_{i \in J_m} \bar{\boldsymbol{q}} \cdot \boldsymbol{z}_i = \bar{\boldsymbol{q}} \cdot \sum_{i \in J_m} \boldsymbol{z}_i = \bar{\boldsymbol{q}} \cdot \boldsymbol{y} = \bar{\boldsymbol{q}}_m \cdot \boldsymbol{x}_m.$$

Since this is true for every z in formula (16), we conclude that $\bar{\boldsymbol{q}}_m \cdot \boldsymbol{x}_m \ge \bar{\boldsymbol{p}}_m \cdot \boldsymbol{x}_m$. Observing that this applies to every m, we complete the proof by the calculation

$$\bar{\boldsymbol{q}} \cdot \boldsymbol{x} = \sum_{m=1}^{M} \bar{\boldsymbol{q}}_m \cdot \boldsymbol{x}_m \ge \sum_{m=1}^{M} \bar{\boldsymbol{p}}_m \cdot \boldsymbol{x}_m = \bar{\boldsymbol{p}} \cdot \boldsymbol{x}.$$

□

Lemma 6 *The following statements are true for every agent $i \in I$:*

(a) $\boldsymbol{x} \in \boldsymbol{P}_i(\bar{\boldsymbol{x}}_i)$ implies $\bar{\boldsymbol{p}} \cdot \boldsymbol{x} \ge \bar{\boldsymbol{p}} \cdot \bar{\boldsymbol{x}}_i$,

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(b) $\bar{\boldsymbol{p}} \cdot \bar{\boldsymbol{x}}_i = \bar{\boldsymbol{p}} \cdot \bar{\boldsymbol{y}}_i$, *and*
(c) $\boldsymbol{y} \in \boldsymbol{Y}_i$ *implies* $\bar{\boldsymbol{p}} \cdot \boldsymbol{y} \leq \bar{\boldsymbol{p}} \cdot \bar{\boldsymbol{y}}_i$.

Proof (a) Since $\boldsymbol{x}$ belongs to the closure of the interior of $\hat{\boldsymbol{P}}_i(\bar{\boldsymbol{x}}_i)$, part (a) of Lemma 3 implies that $\boldsymbol{p}_i \cdot \boldsymbol{x} \geq \boldsymbol{p}_i \cdot \bar{\boldsymbol{x}}_i$. Defining $\boldsymbol{x}' = \boldsymbol{x} \wedge \bar{\boldsymbol{x}}_i \in \boldsymbol{X}_i \cap \boldsymbol{K}$, we have $\boldsymbol{p}_i \cdot (\boldsymbol{x} - \boldsymbol{x}') \geq \boldsymbol{p}_i \cdot (\bar{\boldsymbol{x}}_i - \boldsymbol{x}')$. Using this result, Lemma 5, the inequality in (15), and part (b) of Lemma 4, we calculate that

$$\begin{aligned}\bar{\boldsymbol{p}} \cdot (\boldsymbol{x} - \bar{\boldsymbol{x}}_i) + \bar{\boldsymbol{q}} \cdot (\bar{\boldsymbol{x}}_i - \boldsymbol{x}') &\geq \bar{\boldsymbol{p}} \cdot (\boldsymbol{x} - \bar{\boldsymbol{x}}_i) + \bar{\boldsymbol{p}} \cdot (\bar{\boldsymbol{x}}_i - \boldsymbol{x}') = \bar{\boldsymbol{p}} \cdot (\boldsymbol{x} - \boldsymbol{x}') \\ &\geq \boldsymbol{p}_i \cdot (\boldsymbol{x} - \boldsymbol{x}') \geq \boldsymbol{p}_i \cdot (\bar{\boldsymbol{x}}_i - \boldsymbol{x}') = \bar{\boldsymbol{q}} \cdot (\bar{\boldsymbol{x}}_i - \boldsymbol{x}').\end{aligned}$$

This yields $\bar{\boldsymbol{p}} \cdot \boldsymbol{x} \geq \bar{\boldsymbol{p}} \cdot \bar{\boldsymbol{x}}_i$, as required.

(b) We first use Lemma 5, the restricted quasi-equilibrium properties, part (b) of Lemma 4, and (15) to obtain the inequality

$$\bar{\boldsymbol{p}} \cdot \bar{\boldsymbol{y}}_i \leq \bar{\boldsymbol{q}} \cdot \bar{\boldsymbol{y}}_i = \bar{\boldsymbol{q}} \cdot \bar{\boldsymbol{x}}_i \leq \bar{\boldsymbol{p}}_i \cdot \bar{\boldsymbol{x}}_i \leq \bar{\boldsymbol{p}} \cdot \bar{\boldsymbol{x}}_i. \tag{17}$$

On the other hand, we utilize part (a) to deduce that $\bar{\boldsymbol{p}}$ is positive, and it follows that $\sum_{i \in I} (\bar{\boldsymbol{p}} \cdot \bar{\boldsymbol{y}}_i - \bar{\boldsymbol{p}} \cdot \bar{\boldsymbol{x}}_i) \geq 0$. Together these inequalities yield the desired result.

(c) Observing that $\boldsymbol{y} \in \boldsymbol{K}_+$, we combine Lemma 5, the restricted quasi-equilibrium properties, inequalities in (17), and part (b) of this lemma as follows:

$$\bar{\boldsymbol{p}} \cdot \boldsymbol{y} \leq \bar{\boldsymbol{q}} \cdot \boldsymbol{y} \leq \bar{\boldsymbol{q}} \cdot \bar{\boldsymbol{y}}_i \leq \bar{\boldsymbol{p}} \cdot \bar{\boldsymbol{x}}_i = \bar{\boldsymbol{p}} \cdot \bar{\boldsymbol{y}}_i.$$

This completes the proof. □

By property (14) and the right hand side of the equality sign in (17), there exists an agent i with $\bar{\boldsymbol{p}} \cdot \bar{\boldsymbol{x}}_i \geq \bar{\boldsymbol{q}} \cdot \bar{\boldsymbol{x}}_i > 0$. This observation and Lemma 6 finalize the proof of the theorem.

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