A Limit Theorem for Financial Markets with Inert Investors

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We study the effect of investor inertia on stock price fluctuations with a market microstructure model comprising many small investors who are inactive most of the time. It turns out that semi-Markov processes are tailor made for modelling inert investors. With a suitable scaling, we show that when the price is driven by the market imbalance, the log price process is approximated by a process with long range dependence and non-Gaussian returns distributions, driven by a fractional Brownian motion. Consequently, investor inertia may lead to arbitrage opportunities for sophisticated market participants. The mathematical contributions are a functional central limit theorem for stationary semi-Markov processes, and approximation results for stochastic integrals of continuous semimartingales with respect to fractional Brownian motion.

Key words: Semi-Markov processes; fractional Brownian motion; functional central limit theorem; market microstructure; investor inertia.

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1. Introduction and Motivation We prove a functional central limit theorem for stationary semi-Markov processes in which the limit process is a stochastic integral with respect to fractional Brownian motion. Our motivation is to develop a probabilistic framework within which to analyze the aggregate effect of investor inertia on asset price dynamics. We show that, in isolation, such infrequent trading patterns can lead to long-range dependence in stock prices and arbitrage opportunities for other more "sophisticated" traders.

1.1 Market Microstructure Models for Financial Markets In mathematical finance, the dynamics of asset prices are usually modelled by trajectories of some exogenously specified stochastic process defined on some underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Geometric Brownian motion has long become the canonical reference model of financial price fluctuations. Since prices are generated by the demand of market participants, it is of interest to support such an approach by a microeconomic model of interacting agents.

In recent years there has been increasing interest in agent-based models of financial markets. These models are capable of explaining, often through simulations, many facts like the emergence of herding behavior [41], volatility clustering [42] or fat-tailed distributions of stock returns [17] that are observed in financial data. Brock and Hommes [10, 11] proposed models with many traders where the asset price process is described by *deterministic* dynamical systems. From numerical simulations, they showed that financial price fluctuations can exhibit chaotic behavior if the effects of technical trading become too strong.

Föllmer and Schweizer [27] took the probabilistic point of view, with asset prices arising from a sequence of temporary price equilibria in an exogenous random environment of investor sentiment; see [25], [32] or [26] for similar approaches. Applying an invariance principle to a sequence of suitably defined discrete time models, they derived a diffusion approximation for the logarithmic price process. Duffie and Protter [22] also provided a mathematical framework for approximating sequences of stock prices by diffusion processes.

All the aforementioned models assume that the agents trade the asset in each period. At the end of each trading interval, the agents update their expectations for the future evolution of the stock price and formulate their excess demand for the following period. However, small investors are not so efficient in their investment decisions: they are typically inactive and actually trade only occasionally. This may be because they are waiting to accumulate sufficient capital to make further stock purchases; or they tend to monitor their portfolios infrequently; or they are simply scared of choosing the wrong investments; or they feel that as long-term investors, they can defer action; or they put off the time-consuming research necessary to make informed portfolio choices. Long uninterrupted periods of inactivity may be viewed as a form of investor inertia. The focus of this paper is the effect of such investor inertia on asset prices in a model with asynchronous order arrivals. See [37] for an alternative micro-structure model with asynchronous trading.

1.2 Inertia in Financial Markets Investor inertia is a common experience and is well documented. The New York Stock Exchange (NYSE)'s survey of individual shareownership in the United States, "Shareownership2000" [46], demonstrates that many investors have very low levels of trading activity. For example they find that "23 percent of stockholders with brokerage accounts report no trading at all, while 35 percent report trading only once or twice in the last year" (see pages 58-59). The NYSE survey (e.g. Table 28) also reports that the average holding period for stocks is long, for example 2.9 years in the early 90's.

Empirical evidence of inertia also appears in the economic literature. For example, Madrian and Shea [43] looked at the reallocation of assets in employees' individual 401(k) (retirement) plans[1] and found "a status quo bias resulting from employee procrastination in making or implementing an optimal savings decision." A related study by Hewitt Associates (a management consulting firm) found that in 2001, four out of five plan participants did not do any trading in their 401(k)s. Madrian and Shea explain that "if the cost of gathering and evaluating the information needed to make a 401(k) savings decision exceeds the short-run benefit from doing so, individuals will procrastinate." The prediction of Prospect Theory [35] that investors tend to hold onto losing stocks too long has also been observed ([50]).

A number of microeconomic models study investor caution with regard to model risk, which is termed uncertainty aversion. Among others, Dow and Werlang ([21]) and Simonsen and Werlang ([51]) considered models of portfolio optimization where agents are uncertain about the true probability measure. Their investors maximize their utility with respect to nonadditive probability measures. It turns out that uncertainty aversion leads to inertia: the agents do not trade the asset unless the price exceeds or falls below a certain threshold.

We provide a mathematical framework for modelling investor inertia in a simple microstructure model where asset prices result from the demand of a large number of small investors whose trading behavior exhibits inertia. To each agent a, we associate a stationary semi-Markov process $x^a = (x^a_t)_{t\geq 0}$ on a finite state space which represents the agent's propensity for trading. The processes x^a have heavy-tailed sojourn times in some designated "inert" state, and relatively thin-tailed sojourn times in various other states. Semi-Markov processes are tailor made to model individual traders' inertia as they generalize Markov processes by removing the requirement of exponentially distributed, and therefore thin-tailed, holding times. In addition, we allow for a market-wide amplitude process Ψ, that describes the evolution of typical trading *size* in the market. It is large on heavy-trading days and small on light trading days. We adopt a non-Walrasian approach to asset pricing and assume that prices move in the direction of market imbalance. We show that in a model with many inert investors, long range dependence in the price process emerges.

1.3 Long Range Dependence in Financial Time Series The observation of long range dependence (sometimes called the Joseph effect) in financial time series motivated the use of fractional Brownian motion as a basis for asset pricing models; see, for instance, [44] or [19]. By our invariance principle, the drift-adjusted logarithmic price process converges weakly to a stochastic integral with respect to a fractional Brownian motion with Hurst coefficient $H > \frac{1}{2}$. Our approach may thus be viewed as a microeconomic foundation for these models. A recent paper that proposes entirely different economic foundations for models based on fractional Brownian motion is [36]. An approximation result for

[1] A 401k retirement plan is a special type of account funded through pre-tax payroll deductions. The funds in the account can be invested in a number of different stocks, bonds, mutual funds or other assets, and are not taxed on any capital gains, dividends, or interest until they are withdrawn. The retirement savings vehicle was created by United States Congress in 1981 and gets its name from the section of the Internal Revenue Code that describes it.

fractional Brownian motion in the context of a binary market model is given in [52].

As is well known, fractional Brownian motion processes are not semimartingales, and so these models may theoretically allow arbitrage opportunities. Explicit arbitrage strategies for various models were constructed in [49], [13] and [3]. These strategies capitalize on the smoothness of fractional Brownian motion (relative to standard Brownian motion) and involve rapid trading to exploit the fine-scale properties of the process' trajectories. As a result, in our microstructure model, arbitrage opportunities may arise for other, sufficiently sophisticated, market participants who are able to take advantage of inert investors by trading frequently. We discuss a simple combination of both inert and active traders in Section 2.3.

Evidence of long-range dependence in financial data is discussed in [19]. Bayraktar *et al.* [5] studied an asymptotically efficient wavelet-based estimator for the Hurst parameter, and analyzed high frequency S&P 500 index data over the span of 11.5 years (1989-2000). It was observed that, although the Hurst parameter was significantly above the efficient markets value of $H = \frac{1}{2}$ up through the mid-1990s, it started to fall to that level over the period 1997-2000 (see Figure 1). They suggested that this behavior of the market might be related to the increase in Internet trading, which is documented, for example, in NYSE's Stockownership2000 [46], [1], and [14], who find that "after 18 months of access, the Web effect is very large: trading frequency doubles." Barber and Odean [2] find that "after going online, investors trade more actively, more speculatively and less profitably than before". Similar empirical findings were recently reached, using a completely different statistical technique in [6]. Thus, the dramatic fall in the estimated Hurst parameter in the late 1990s can be thought of as *a posteriori* validation of the link our model provides between investor inertia and long-range dependence in stock prices.

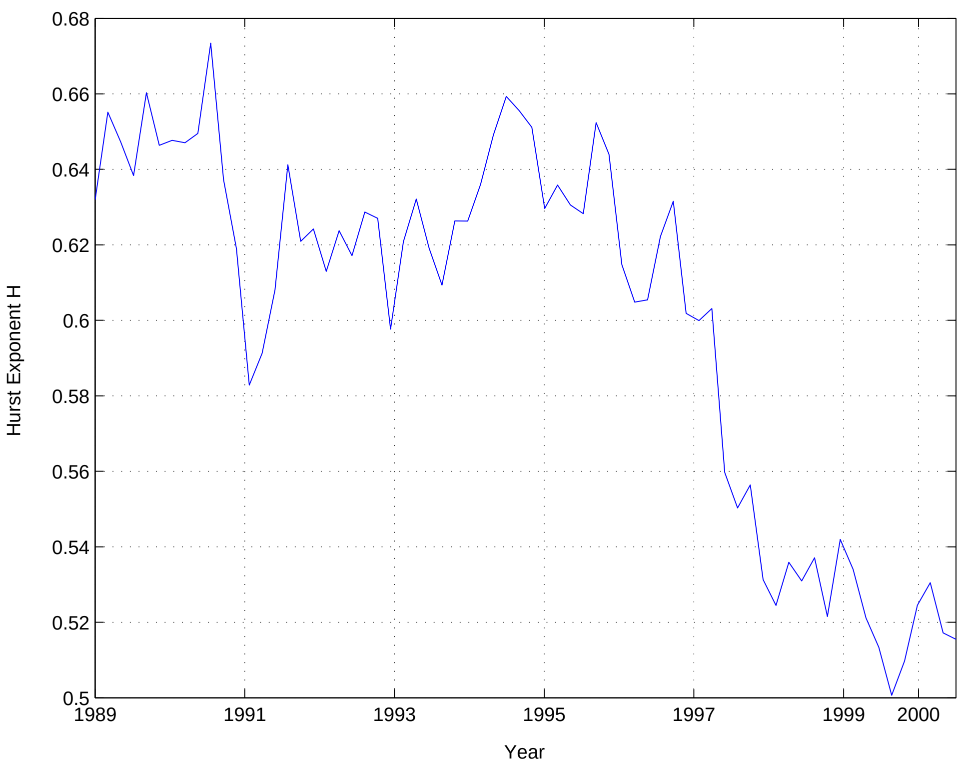

Figure 1: *Estimates of the Hurst exponent of the S&P 500 index over 1990s, taken from Bayraktar, Poor and Sircar ([5]).*

We note the evidence of long memory in stock price returns is mixed. There are several papers in the empirical finance literature providing evidence for the existence of long memory, yet there are several other papers that contradict these empirical findings; see e.g. [5] for an exposition of this debate and references. However, long memory is a well accepted feature in volatility (squared and absolute returns) and trading volume (see e.g. [18] and [20]). The mathematical results of this paper might also be seen as an intermediate step towards a microstructural foundation for this phenomenon.

1.4 Mathematical Contributions We establish a functional central limit theorem for semi-Markov processes (Theorem 2.1 below) which extends the results of Taqqu et. al. [54], who proved a result similar to ours for on/off processes, that is, semi-Markov processes taking values in the binary state space $\{0,1\}$. Their arguments do not carry over to models with more general state spaces. Our approach builds on Markov renewal theory. We also demonstrate (see Example 3.1) that there may be a different limit behavior when the semi-Markov processes are centered, a situation which cannot arise in the binary case. Taqqu and Levy [53] considered renewal reward processes with heavy tailed renewal periods and independent and identically distributed rewards. They assume a general state space, but the distributions of the length of renewal periods does not depend on the current state; for an extension to the case of heavy-tailed rewards, see [40]. A recent paper [45] studies the binary case under a different limit taking mechanism; see also [28].

Binary state spaces are natural for modelling internet traffic, but for many applications in Economics or Queueing Theory, it is clearly desirable to have more flexible results that apply to general semi-Markov processes on finite state spaces. In the context of a financial market model, it is natural to allow for both positive (buying), negative (selling) and a zero (inactive) state. Our results also have applications to complex multi-level queueing networks where the level-dependent holding-time distributions are allowed to have slowly decaying tails. They may serve as a mathematical basis for proving heavy-traffic limits in the network models studied in, e.g. [23], [24] and [55].

We allow for limits which are integrals with respect to fractional Brownian motion proving an approximation result for stochastic integrals of continuous semimartingales with respect to fractional Brownian motion. Specifically, we consider a sequence of good semimartingales $\{\Psi^n\}$ and a sequence of stochastic processes $\{X^n\}$ having zero quadratic variation and give sufficient conditions which guarantee that joint convergence of (X^n, Ψ^n) to (B^H, Ψ), where B^H is a fractional Brownian motion process with Hurst parameter $H > \frac{1}{2}$, and Ψ is a continuous semimartingale, implies the convergence of the stochastic integrals $\int \Psi^n dX^n$ to $\int \Psi dB^H$. In addition, we obtain a stability result for the integral of a fractional Brownian motion with respect to itself. These results may be viewed as an extension of Theorem 2.2 in [38] beyond the semimartingale setting.

The remainder of this paper is organized as follows. In Section 2, we describe the financial market model with inert investors and state the main result. Section 3 proves a central limit theorem for stationary semi-Markov processes. Section 4 proves an approximation result for stochastic integrals of continuous semimartingales with respect to fractional Brownian motion.

2. The microeconomic setup and the main results We consider a financial market with a set $\mathbb{A} := \{a_1, a_2, \ldots, a_N\}$ of *agents* trading a single risky asset. Our aim is to analyze the effects investor inertia has on the dynamics of stock price processes. For this we choose the simplest possible setup. In particular, we model right away the behavior of individual traders rather than characterizing agents' investment decisions as solutions to individual utility maximization problems. Such an approach has also been taken in [29], [27], [41], [26] and [37] for example.

We associate to each agent $a \in \mathbb{A}$ a continuous-time stochastic process $x^a = (x^a_t)_{t \geq 0}$ on a finite state space E, containing zero. This process describes the agent's *trading mood*. He accumulates the asset at a rate $\Psi_t x^a_t$ at time $t \geq 0$. The random quantity $\Psi_t > 0$ describes the size of a typical trade at time t, and x^a_t may be negative, indicating the agent is selling. Agents do not trade at times when $x^a_t = 0$. We therefore call the state 0 the agents' *inactive state*.

Remark 2.1 *In the simplest setting, $x^a \in \{-1, 0, 1\}$, so that each investor is either buying, selling or inactive, and $\Psi \equiv 1$: there is no external amplification. Even here, the existing results in [54] do not apply because the state space is not binary.*

The holdings of the agent $a \in \mathbb{A}$ and the "market imbalance" at time $t \geq 0$ are given by

$$\int_0^t \Psi_s x^a_s ds \qquad \text{and} \qquad I^N_t := \sum_{a \in \mathbb{A}} \int_0^t \Psi_s x^a_s ds, \tag{1}$$

respectively. Hence the process $(I^N_t)_{t \geq 0}$ describes the stochastic evolution of the *market imbalance*. In our microstructure model, market imbalance will be the only component driving the dynamics of asset prices. All the orders are received by a single market maker who clears the trades and sets prices as to reflect the incoming order flows. That is, the market maker sets prices in reaction to the evolution of market imbalances.

REMARK 2.2 *In our continuous time model buyers and sellers arrive at different points in time. Hence the economic paradigm that a Walrasian auctioneer can set prices such that the markets clear at the end of each trading period does not apply. Rather, temporary imbalances between demand and supply will occur, and prices are assumed to reflect the extent of the current market imbalance. In the terminology explained in [29], ours is a model of a "continuous market (trading asynchronously during continuous intervals of time)", rather than a "call market (trading synchronously at pre-established discrete times)". As Garman reports, the New York Stock Exchange was a call market until 1871, and since then has become a continuous market. (See also Chapter 1 of [47].)*

We consider the pricing rule

$$dS_t^N = \sum_{a \in \mathbb{A}} \Psi_t x_t^a dt \quad \text{and so} \quad S_t^N = S_0 + I_t^N, \tag{2}$$

for the evolution of the logarithmic stock price process $S^N = (S_t^N)_{t \geq 0}$. This is the simplest mechanism by which incoming buy orders increase the price and sell orders decrease the price. Other choices might be utilized in a future work studying, for example, the effect of a nonlinear market depth function, but these are beyond the scope of the present work. (The choice of modelling the log-stock price is simply standard finance practice to define a positive price).

Kruk ([37]) considered a model for continuous auction market, in which order arrivals are modelled by independent renewal processes. There are a finite number of possible prices, and agents randomly submit price dependent limit orders. These are stored in the order book waiting the arrival of matching orders. Kruk finds a limiting distribution of the outstanding number of buy/sell orders at one of the possible prices. In contrast, our aim is to find the limiting price process that is driven by the market imbalance under different assumptions on the market micro-structure.

2.1 The dynamics of individual behavior Next, we specify the probabilistic structure of the processes x^a. We assume that the agents are homogeneous and that all the processes x^a and Ψ are independent. It is therefore enough to specify the dynamics of some reference process $x = (x_t)_{t \geq 0}$. In order to incorporate the idea of market inertia as defined by Assumption 2.2 below, we assume that x is a *semi-Markov* process defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a finite state space E. Here E may contain both positive and negative values and we assume $0 \in E$. The process x is specified in terms of random variables $\xi_n : \Omega \to E$ and $T_n : \Omega \to \mathbb{R}_+$ which satisfy $0 = T_0 \leq T_1 \leq \cdots$ almost surely and

$$\mathbb{P}\{\xi_{n+1} = j, T_{n+1} - T_n \leq t \big| \xi_1, ..., \xi_n; T_1, ..., T_n\} = \mathbb{P}\{\xi_{n+1} = j, T_{n+1} - T_n \leq t \big| \xi_n\}$$

for each $n \in \mathbb{N}$, $j \in E$ and all $t \in \mathbb{R}_+$ through the relation

$$x_t = \sum_{n \geq 0} \xi_n \mathbf{1}_{[T_n, T_{n+1})}(t). \tag{3}$$

REMARK 2.3 *In economic terms, the representative agent's mood in the random time interval $[T_n, T_{n+1})$ is given by ξ_n. The distribution of the length of the interval $T_{n+1} - T_n$ may depend on the sequence $\{\xi_n\}_{n \in \mathbb{N}}$ through the states ξ_n and ξ_{n+1}. This allows us to assume different distributions for the lengths of the agents' active and inactive periods, and in particular to model inertia as a heavy-tailed sojourn time in the zero state.*

REMARK 2.4 *In the present analysis of investor inertia, we do not allow for feedback effects of prices into agents' investment decisions. While such an assumption might be justified for small, non-professional investors, it is clearly desirable to allow active traders' investment decisions to be influenced by asset prices. When such feedback effects are allowed, the analysis of the price process is typically confined to numerical simulations because such models are difficult to analyze on an analytical level. An exception is a recent paper [26] where the impact of contagion effects on the asymptotics of stock prices is analyzed in a mathematically rigorous manner. One could also consider the present model as applying to (Internet* *or new economy) stocks where no accurate information about the actual underlying fundamental value is available. In such a situation, price is not always a good indicator of value and is often ignored by uninformed small investors.*

We assume that x is temporally homogeneous under the measure $\mathbb{P}$, that is,

$$\mathbb{P}\{\xi_{n+1} = j,\ T_{n+1} - T_n \leq t \big| \xi_n = i\} = Q(i,j,t) \tag{4}$$

is independent of $n \in \mathbb{N}$. By Proposition 1.6 in [15], this implies that $\{\xi_n\}_{n\in\mathbb{N}}$ is a homogeneous Markov chain on E whose transition probability matrix $P = (p_{ij})$ is given by

$$p_{ij} = \lim_{t\to\infty} Q(i,j,t).$$

Clearly, x is an ordinary temporally homogeneous Markov process if Q takes the form

$$Q(i,j,t) = p_{ij}\left(1 - e^{-\lambda_i t}\right). \tag{5}$$

We assume that the *embedded Markov chain* $\{\xi_n\}_{n\in\mathbb{N}}$ satisfies the following condition.

Assumption 2.1 *For all $i,j \in E$, $i \neq j$ we have that $p_{ij} > 0$. In particular, there exists a unique probability measure π on E such that $\pi P = \pi$.*

The conditional distribution function of the length of the n-th sojourn time, $T_{n+1} - T_n$, given ξ_{n+1} and ξ_n is specified in terms of the *semi-Markov kernel* $\{Q(i,j,t); i,j \in E,\ t \geq 0\}$ and the transition matrix P by

$$G(i,j,t) := \frac{Q(i,j,t)}{p_{ij}} = \mathbb{P}\{T_{n+1} - T_n \leq t | \xi_n = i,\ \xi_{n+1} = j\}. \tag{6}$$

For later reference we also introduce the distribution of the first occurrence of state j under $\mathbb{P}$, given $x_0 = i$. Specifically, for $i \neq j$, we put

$$F(i,j,t) := \mathbb{P}\{\tau_j \leq t | x_0 = i\}, \tag{7}$$

where $\tau_j := \inf\{t \geq 0 : x_t = j\}$. We denote by $F(j,j,\cdot)$ the distribution of the time until the next entrance into state j and by

$$\eta_j := \int tF(j,j,dt) \tag{8}$$

the expected time between two occurrences of state $j \in E$. Further, we recall that a function $L : \mathbb{R}_+ \to \mathbb{R}_+$ is called *slowly varying at infinity* if

$$\lim_{t\to\infty} \frac{L(xt)}{L(t)} = 1 \qquad \text{for all} \qquad x > 0$$

and that $f(t) \sim g(t)$ for two functions $f, g : \mathbb{R}_+ \to \mathbb{R}_+$ means $\lim_{t\to\infty} \frac{f(t)}{g(t)} = 1$.

Assumption 2.2 *(i) The average sojourn time at state $i \in E$ is finite:*

$$m_i := \mathbb{E}[T_{n+1} - T_n | \xi_n = i] < \infty. \tag{9}$$

Here $\mathbb{E}$ denotes the expectation operator with respect to $\mathbb{P}$.

(ii) There exists a constant $1 < \alpha < 2$ and a locally bounded function $L : \mathbb{R}_+ \to \mathbb{R}_+$ which is slowly varying at infinity such that

$$\mathbb{P}\{T_{n+1} - T_n \geq t \big| \xi_n = 0\} \sim t^{-\alpha} L(t). \tag{10}$$

(iii) The distributions of the sojourn times at state $i \neq 0$ satisfy

$$\lim_{t\to\infty} \frac{\mathbb{P}\{T_{n+1} - T_n \geq t \big| \xi_n = i\}}{t^{-(\alpha+1)} L(t)} = 0.$$

(iv) The distribution of the sojourn times in the various states have continuous and bounded densities with respect to Lebesgue measure on $\mathbb{R}_+$.

Our condition (10) is satisfied if, for instance, the length of the sojourn time at state $0 \in E$ is distributed according to a Pareto distribution. Assumption 2.2 (iii) reflects the idea of market inertia: the probability of long uninterrupted trading periods is small compared to the probability of an individual agent being inactive for a long time. It is stronger than the corresponding assumption for the binary case in [54] where the sojourn time in the only other state may in fact be as heavy tailed. For our economic application, however, it is natural to think of the sojourn times in the various active states as being thin tailed, such as in the exponential distribution, since small investors typically do not trade continually for long periods.

2.2 An invariance principle for semi-Markov processes In this section, we state our main results. With our choice of scaling, the logarithmic price process can be approximated in law by the stochastic integral of Ψ with respect to fractional Brownian motion B^H where the *Hurst coefficient* H depends on α. The convergence concept we use is weak convergence on the Skorohod space $\mathbb{D}$ of all real-valued right continuous processes with left limits. We write $\mathcal{L}\text{-}\lim_{n\to\infty} Z^n = Z$ if a sequence of $\mathbb{D}$-valued stochastic processes $\{Z^n\}_{n\in\mathbb{N}}$, converges in distribution to Z.

In order to derive our approximation result, we assume that the semi-Markov process x is stationary. Under Assumption 2.1, stationarity can be achieved by a suitable specification of the common distribution of the initial state ξ_0 and the initial sojourn time T_1. We denote the distribution of the stationary semi-Markov processes by $\mathbb{P}^*$. The proof follows from Theorem 4.2.5 in [9], for example.

Lemma 2.1 *In the stationary setting, that is, under the law $\mathbb{P}^*$ the following holds:*

(i) The joint distribution of the initial state and the initial sojourn time takes the form

$$\mathbb{P}^*\{\xi_0 = k, T_1 > t\} = \frac{\pi_k}{\sum_{j\in E}\pi_j m_j}\int_t^\infty h(k,s)ds. \tag{11}$$

Here m_i denotes the mean sojourn time in state $i \in E$ as defined by (9), and for $i \in E$,

$$h(i,t) = 1 - \sum_{j\in E} Q(i,j,t) \tag{12}$$

is the probability that the sojourn time at state $i \in E$ is greater than t.

(ii) The law $\nu = (\nu_k)_{k\in E}$ of x_t in the stationary regime is given by

$$\nu_k = \frac{\pi_k m_k}{\sum_{j\in E}\pi_j m_j}. \tag{13}$$

(iii) The conditional joint distribution of (ξ_1, T_1), given ξ_0 is

$$\mathbb{P}^*\{\xi_1 = j, T_1 < t \mid \xi_0 = k\} = \frac{p_{kj}}{m_{k,j}}\int_0^t [1 - G(k,j,s)]ds. \tag{14}$$

Here $m_{k,j} := \int_0^\infty [1-G(k,j,s)]ds$ denotes the conditional expected sojourn time at state k, given the next state is j, and the functions $G(k,j,\cdot)$ are defined in (6).

Let us now introduce a dimensionless parameter $\varepsilon > 0$, and consider the rescaled processes $x^a_{t/\varepsilon}$. For ε small, $x^a_{t/\varepsilon}$ is a "speeded-up" semi-Markov process. In other words, the investors' individual trading dispensations are evolving on a faster scale than Ψ. Observe, however, that we are not altering the main qualitative feature of the model. That is, agents still remain in the inactive state for relatively much longer times than in an active state.

Mathematically, there is no reason to restrict ourselves to the case where Ψ is non-negative. Hence we shall from now on only assume that Ψ is a continuous semimartingale. Given the processes Ψ and x^a ($a \in \{a_1, \ldots, a_N\}$), the aggregate order rate at time t is given by

$$Y^{\varepsilon,N}_t = \sum_{a\in\mathbb{A}} \Psi_t x^a_{t/\varepsilon}. \tag{15}$$

Let $\mu := \mathbb{E}^* x_t$ and $X^{\varepsilon,N} = (X^{\varepsilon,N}_t)_{0\le t\le T}$ ($T > 0$) be the centered process defined by

$$X^{\varepsilon,N}_t := \int_0^t \sum_{a\in\mathbb{A}} \Psi_s(x^a_{s/\varepsilon} - \mu)\,ds = \int_0^t Y^{\varepsilon,N}_s\,ds - \mu N\int_0^t \Psi_s ds. \tag{16}$$

We are now ready to state our main result. Its proof will be carried out in Sections 3 and 4. The definition of the stochastic integral with respect to fractional Brownian motion will also be given in Section 4.

Theorem 2.1 *Let $\Psi = (\Psi_t)_{t\geq 0}$ be a continuous semimartingale on $(\Omega, \mathcal{F}, \mathbb{P}^*)$ with a decomposition $\Psi = M + A$, in which M is a local martingale and A is an adapted process of finite variation. We assume that $\mathbb{E}\{[M,M]_T\} < \infty$ and that $\mathbb{E}\{|A|_T\} < \infty$, where $(|A|_t)_{t\geq 0}$ is the total variation of A. If Assumptions 2.1 and 2.2 are satisfied, and if $\mu \sum_{k\in E} k \frac{m_k}{\eta_k^2} > 0$, then there exists $c > 0$ such that the process $X^{\varepsilon,N}$ satisfies*

$$\mathcal{L}\text{-}\lim_{\varepsilon\downarrow 0} \mathcal{L}\text{-}\lim_{N\to\infty} \left(\frac{1}{\varepsilon^{1-H}\sqrt{NL(\varepsilon^{-1})}} X_t^{\varepsilon,N} \right)_{0\leq t\leq T} = \left(c \int_0^t \Psi_s dB_s^H \right)_{0\leq t\leq T}. \tag{17}$$

Here the Hurst coefficient of the fractional Brownian motion process B^H is $H = \frac{3-\alpha}{2} > \frac{1}{2}$.

Observe that Theorem 2.1 does not apply to the case $\mu = 0$. For centered semi-Markov processes x^a, Example 3.1 below illustrates that the limiting process depends on the tail structure of the waiting time distribution in the various active states. This phenomenon does not arise in the case of binary state spaces.

Remark 2.5 *(i) Theorem 2.1 says the drift-adjusted logarithmic price process in our model of inert investors can be approximated in law by the stochastic integral of Ψ with respect to a fractional Brownian motion process with Hurst coefficient $H > \frac{1}{2}$.*

(ii) In a situation where the processes x^a are independent, stationary and ergodic Markov processes on E, that is, in cases where the semi-Markov kernel takes the form (5), it is easy to show that

$$\mathcal{L}\text{-}\lim_{\varepsilon\downarrow 0} \mathcal{L}\text{-}\lim_{N\to\infty} \left(\frac{1}{\sqrt{\varepsilon N}} X_t^{\varepsilon,N} \right)_{0\leq t\leq T} = \left(c \int_0^t \Psi_s dW_s \right)_{0\leq t\leq T}$$

where $(W_t)_{t\geq 0}$ is a standard Wiener process. Thus, if the market participants are not inert, that is, if the distribution of the lengths of the agents' inactivity periods is thin-tailed, no arbitrage opportunities emerge because the limit process is a semimartingale.

The proof of Theorem 2.1 will be carried out in two steps. In Section 3 we prove a functional central limit theorem for stationary semi-Markov processes on finite state spaces. In Section 4 we combine our central limit theorem for semi-Markov processes with extensions of arguments given in [38] to obtain (17).

2.3 Markets with both Active and Inert Investors It is simple to extend the previous analysis to incorporate both active and inert investors. Let ρ be the ratio of active to inert investors. We associate to each *active* trader $b \in \{1, 2, \ldots, \rho N\}$ a stationary Markov chain $y^b = (y_t^b)_{t\geq 0}$ on the state space E. The processes y^b are independent and identically distributed and independent of the processes x^a. The thin-tailed sojourn time in the zero state of y^b reflects the idea that, as opposed to inert investors, these agents frequently trade the stock. We assume for simplicity that $\Psi \equiv 1$. With $\hat{Y}_t^{\varepsilon,N} = \sum_{b=1}^{\rho N} \left(y_{t/\varepsilon}^b - \mathbb{E}^* y_0 \right)$ and $\hat{X}_t^{\varepsilon,N} := \int_0^t \hat{Y}_s^{\varepsilon,N} ds$, it is straightforward to prove the following modification of Theorem 2.1.

Proposition 2.1 *Let x^a $(a = 1, 2, \ldots, N)$ be semi-Markov processes that satisfy the assumption of Theorem 2.1. If y^b $(b = 1, 2, \ldots, \rho N)$ are independent stationary Markov processes on E, then there exist constants $c_1, c_2 > 0$ such that*

$$\mathcal{L}\text{-}\lim_{\varepsilon\downarrow 0} \mathcal{L}\text{-}\lim_{N\to\infty} \left(\frac{1}{\varepsilon^{1-H}\sqrt{NL(\varepsilon^{-1})}} X_t^{\varepsilon,N} + \frac{1}{\sqrt{N\varepsilon}} \hat{X}_t^{\varepsilon,N} \right)_{0\leq t\leq T} = \left(c_1 B_t^H + c_2\sqrt{\rho}\, W_t \right)_{0\leq t\leq T}.$$

Here, $W = (W_t)_{t\geq 0}$ is a standard Wiener process.

Thus, in a financial market with both active and inert investors, the dynamics of the asset price process can be approximated in law by a stochastic integral with respect to a superposition, $B^H + \delta W$, of a fractional and a regular Brownian motion. It is known ([12]) that $B^H + \delta W$ is a semimartingale for any $\delta \neq 0$, if $H > \frac{3}{4}$, that is, if $\alpha < \frac{3}{2}$, but not if $H \in (\frac{1}{2}, \frac{3}{4}]$. Thus, no arbitrage opportunities arise if the small investors are "sufficiently inert." The parameter α can also be viewed as a measure for the fraction of small investors that are active at any point in time. Hence, independent of the actual trading volume, the market is arbitrage free in periods where the fraction of inert investors who are active on the financial market is small enough.

3. A limit theorem for semi-Markov processes This section establishes Theorem 2.1 for the special case $\Psi \equiv 1$. We approach the general case where Ψ is a continuous semimartingale in Section 4. Here we consider the situation where

$$Y_t^{\varepsilon,N} = \sum_{a\in\mathbb{A}} x^a_{t/\varepsilon} \quad \text{and where} \quad X_t^{\varepsilon,N} = \int_0^t Y_s^{\varepsilon,N}\,ds - N\mu t,$$

and prove a functional central limit theorem for stationary semi-Markov processes. Our Theorem 3.1 below extends the results in [54] to situations where the semi-Markov process takes values in an arbitrary finite state space. The arguments given there are based on results from ordinary renewal theory, and do not carry over to models with more general state spaces. The proof of the following theorem will be carried out through a series of lemmas.

THEOREM 3.1 *Let $H = \frac{3-\alpha}{2}$. Under the assumptions of Theorem 2.1,*

$$\mathcal{L}\text{-}\lim_{\varepsilon\downarrow 0}\mathcal{L}\text{-}\lim_{N\to\infty}\left(\frac{1}{\varepsilon^{1-H}\sqrt{NL(\varepsilon^{-1})}}X_t^{\varepsilon,N}\right)_{0\le t\le T} = \left(cB_t^H\right)_{0\le t\le T}. \tag{18}$$

Let γ be the covariance function of the semi-Markov process $(x_t)_{t\ge0}$ under $\mathbb{P}^*$, and consider the case $\varepsilon = 1$. By the Central Limit Theorem, and because x is stationary, the process $Y = (Y_t)_{t\ge0}$ defined by

$$Y_t = \mathcal{L}\text{-}\lim_{N\to\infty}\frac{1}{\sqrt{N}}(Y_t^{1,N} - N\mu) \tag{19}$$

is a stationary zero-mean Gaussian process. It is easily checked that the covariance function of the process $(\frac{1}{\sqrt{N}}Y_t^{1,N})$ is also γ for any N, and hence for Y_t. By standard calculations, the variance of the aggregate process $(\int_0^t Y_s\,ds)$ at time $t\ge0$ is given by

$$\mathrm{Var}(t) := \mathrm{Var}\left(\int_0^t Y_s\,ds\right) = 2\int_0^t\left(\int_0^v \gamma(u)du\right)dv. \tag{20}$$

In the first step towards the proof of Theorem 3.1, we can proceed by analogy with [54]. We are interested in the asymptotics as $\varepsilon\downarrow 0$ of the process

$$X_t^\varepsilon := \int_0^t Y_{s/\varepsilon}\,ds, \tag{21}$$

which can be written $X_t^\varepsilon = \varepsilon\int_0^{t/\varepsilon} Y_s\,ds$. Therefore the object of interest is the large t behavior of Var(t). Suppose that we can show

$$\mathrm{Var}(t) \sim c^2 t^{2H} L(t) \quad \text{as} \quad t\to\infty. \tag{22}$$

Then the mean-zero Gaussian processes $X^\varepsilon = (X_t^\varepsilon)_{t\ge0}$ have stationary increments and satisfy

$$\lim_{\varepsilon\downarrow0}\mathbb{E}^*\left(\frac{1}{\varepsilon^{1-H}\sqrt{L(\varepsilon^{-1})}}X_t^\varepsilon\right)^2 = c^2t^{2H}. \tag{23}$$

Since the variance characterizes the finite dimensional distributions of a mean-zero Gaussian process with stationary increments, we see that the finite dimensional distributions of the process $\left(\frac{1}{\varepsilon^{1-H}\sqrt{L(\varepsilon^{-1})}}X_t^\varepsilon\right)_{t\ge0}$ converge to $\left(cB_t^H\right)_{t\ge0}$ whenever (22) holds. The following lemma gives a sufficient condition for (22) in terms of the covariance function γ.

LEMMA 3.1 *For (22) to hold, it suffices that*

$$\gamma(t) \sim c^2H(2H-1)t^{2H-2}L(t) \quad \textit{as} \quad t\to\infty. \tag{24}$$

Proof. By Proposition 1.5.8 in [8], every slowly varying function L which is locally bounded on $\mathbb{R}_+$ satisfies

$$\int_0^t \tau^\beta L(\tau)d\tau \sim \frac{t^{\beta+1}L(t)}{\beta+1}$$

if $\beta > -1$. Applying this proposition to the slowly varying function

$$\tilde{L}(t) := \frac{\gamma(t)}{c^2 H(2H-1)t^{2H-2}},$$

we conclude

$$\int_0^t \int_0^v \gamma(u)du\, dv \sim \frac{c^2}{2}t^{2H}L(t),$$

and so our assertion follows from (20). □

Before we proceed with the proof of our main result, let us briefly consider the case $\mu = 0$ which is not covered by our theorem. For semi-Markov processes whose "heavy-tailed state" happens to be the mean, the structure of the limit process depends on the distribution of the sojourn times in the various active states.[2]

Example 3.1 *We consider the case $E = \{-1, 0, 1\}$, and assume that $p_{-1,0} = p_{1,0} = 1$ and that $p_{0,-1} = p_{0,1} = \frac{1}{2}$. With $\nu_1 = \mathbb{P}^*\{x_t = 1\} > 0$, we obtain*

$$\gamma(t) = \nu_1 \left(\mathbb{E}^*[x_t x_0 | x_0 = 1] + \mathbb{E}^*[x_t x_0 | x_0 = -1]\right).$$

Suppose that the sojourns in the inactive state are heavy tailed, and that the waiting times in the active states are exponentially distributed with parameter 1. In such a symmetric situation

$$\mathbb{E}^*[x_t x_0 | x_0 = \pm 1] = \mathbb{P}^*\{T_1 \geq t | x_0 = \pm 1\} = e^{-t}.$$

Therefore, $\gamma(t) = 2\nu_1 e^{-t}$. In view of (23), this yields $c > 0$ such that

$$\mathcal{L}\text{-}\lim_{\varepsilon \downarrow 0} \mathcal{L}\text{-}\lim_{N\to\infty} \left(\frac{1}{\sqrt{\varepsilon N}} X_t^{\varepsilon,N}\right)_{0\leq t\leq T} = (cW_t)_{0\leq t\leq T}$$

for some standard Wiener process W.

In order to prove Theorem 3.1, we need to establish (24). For this, the following representation of the covariance function turns out to be useful: in terms of the marginal distribution $\nu_i = \mathbb{P}^*\{x_t = i\}$ $(i \in E)$ of the stationary semi-Markov process given in Lemma 2.1 (i), and in terms of the conditional probabilities

$$P_t^*(i,j) := \mathbb{P}^*\{x_t = j | x_0 = i\},$$

we have

$$\gamma(t) = \sum_{i,j\in E} ij\nu_i \left(P_t^*(i,j) - \nu_j\right). \tag{25}$$

It follows from Proposition 6.12 in [15], for example, that $P_t^*(i,j) \to \nu_j$ as $t \to \infty$. Hence $\lim_{t\to\infty} \gamma(t) = 0$. In order to prove Theorem 3.1, however, we also need to show that this convergence is sufficiently slow. We shall see that the agents' inertia accounts for the slow decay of correlations. It is thus the agents' inactivity that is responsible for that fact that the logarithmic price process is not approximated by a stochastic integral with respect to a Wiener process, but by an integral with respect to fractional Brownian motion.

We are now going to determine the rate of convergence of the covariance function to 0. To this end, we show that $P_t^*(i,j)$ can be written as a convolution of a renewal function with a slowly decaying function plus a term which has asymptotically, i.e., for $t \to \infty$, a vanishing effect compared to the first term; see Lemma 3.2 below. We will then apply results from [31] and [34] to analyze the tail structure of the convolution term.

Let

$$R(i,j,t) := \mathbb{E}\left\{\sum_{n=0}^\infty \mathbf{1}_{\{\xi_n = j, T_n \leq t\}} \;\middle|\; x_0 = \xi_0 = i\right\}$$

[2]We thank Chris Rogers for Example 3.1.

be the expected number of visits of the process $(x_t)_{t\geq 0}$ to state j up to time t in the *non-stationary* situation, i.e., under the measure $\mathbb{P}$, given $x_0 = i$. For fixed $i, j \in E$, the function $t \mapsto R(i,j,t)$ is a renewal function. If, under $\mathbb{P}$, the initial state is j, then the entrances to j form an ordinary renewal process and

$$R(j,j,t) = \sum_{n=0}^{\infty} F^n(j,j,t). \tag{26}$$

Here $F(j,j,\cdot)$ denotes the distribution of the travel time between to occurrences of state $j \in E$ as defined in (7), and $F^n(j,j,\cdot)$ is the n-fold convolution of $F(j,j,\cdot)$. On the other hand, if $i \neq j$, the time until the first visit to j has distribution $F(i,j,\cdot)$ under $\mathbb{P}$ which might be different from $F(j,j,\cdot)$. In this case $R(i,j,\cdot)$ satisfies a delayed renewal equation, and we have

$$R(i,j,t) = \int_0^t R(j,j,t-u)F(i,j,du). \tag{27}$$

We refer the interested reader to [15] for a survey on Markov renewal theory.

Let us now return to the stationary setting and derive a representation for the expected number $R^*(i,j,t)$ of visits of the process $(x_t)_{t\geq 0}$ to state j up to time t under $\mathbb{P}^*$, given $x_0 = i$. To this end, we denote by $F^*(i,j,\cdot)$ the distribution function in the stationary setting of the first occurrence of j, given $x_0 = i$ and put

$$P_t(i,j) := \mathbb{P}\{x_t = j | x_0 = i\}.$$

Given the first jump time T_1 and given that $x_{T_1} = i$ we have that

$$\mathbb{P}^*\{x_t = j | x_{T_1} = i\} = P_{t-T_1}(i,j) \quad \text{on} \quad \{t \geq T_1\}. \tag{28}$$

Thus,

$$R^*(i,j,t) = \int_0^t R(j,j,t-u)F^*(i,j,du). \tag{29}$$

3.1 A representation for the conditional transition probabilities In this section we derive a representation for $P_t^*(i,j)$ which will allow us to analyze the asymptotic behavior of $P_t^*(i,j) - \nu_j$. To this end, we recall the definition of the joint distribution of the initial state and the initial sojourn time and the definition of the conditional joint distribution of (ξ_1, T_1), given ξ_0 from (11) and (14) respectively. We define

$$s(i,t) := \mathbb{P}^*\{\xi_0 = i, T_1 > t\} \quad \text{and} \quad \hat{s}(i,j,t) = \mathbb{P}^*\{\xi_1 = j, T_1 \leq t | \xi_0 = i\}. \tag{30}$$

In terms of these quantities, the transition probability $P_t^*(i,j)$ can be written as

$$P_t^*(i,j) = \frac{s(i,t)}{\nu_i}\delta_{ij} + \sum_{k\in E}\int_0^t P_{t-u}(k,j)\hat{s}(i,k,du). \tag{31}$$

Here the first term on the right-hand-side of (31) accounts for the $\mathbb{P}^*$-probability that $x_0 = i$ and that the state i survives until time t. The quantity $\int_0^t P_{t-u}(k,j)\hat{s}(i,k,du)$ captures the conditional probability that the first transition happens to be to state k before time t, given $\xi_0 = i$. Observe that we integrate the conditional probability $P_{t-u}(i,j)$ and not $P_{t-u}^*(i,j)$: conditioned on the value of semi-Markov process at the first renewal instance the distributions of $(x_t)_{t\geq 0}$ under $\mathbb{P}$ and $\mathbb{P}^*$ are the same; see (28).

In the sequel it will be convenient to have the following convolution operation: let $\tilde{h}$ be a locally bounded function, and $\tilde{F}$ be a distribution function both of which are defined on $\mathbb{R}_+$. The convolution $\tilde{F} * \tilde{h}$ of $\tilde{F}$ and $\tilde{h}$ is given by

$$\tilde{F} * \tilde{h}(t) := \int_0^t \tilde{h}(t-x)\tilde{F}(dx) \quad \text{for } t \geq 0. \tag{32}$$

Remark 3.1 *Since $\tilde{F} * \tilde{h}$ is locally bounded, the map $t \mapsto G * (\tilde{F} * \tilde{h})(t)$ is well defined for any distribution G on $\mathbb{R}_+$. Moreover, $G * (\tilde{F} * \tilde{h})(t) = \tilde{F} * (G * \tilde{h})(t) = (G * \tilde{F}) * \tilde{h}(t)$. In this sense distributions acting on the locally bounded function can commute. Thus, for the renewal function $R = \sum_{n=0}^{\infty} \tilde{F}^n$ associated to $\tilde{F}$, as defined in (26), the integral $R * \tilde{h}(t)$ is well defined and $R * (G * \tilde{h})(t) = G * (R * \tilde{h})(t) = (R * G) * \tilde{h}(t)$.*

We are now going to establish an alternative representation for the conditional probability $P^*_t(i,j)$ that turns out to be more appropriate for our subsequent analysis.

LEMMA 3.2 *In terms of the quantities $s(i,t)$ and $h(i,t)$ in (12) and $R^*(i,j,t)$, we have*

$$P^*_t(i,j) = \frac{s(i,t)}{\nu_i}\delta_{ij} + \int_0^t h(j,t-s)R^*(i,j,ds). \tag{33}$$

PROOF. In view of (31), it is enough to show

$$R^*(i,j,t) * h(j,t) = \sum_{k\in E}\int_0^t P_{t-u}(k,j)\hat{s}(i,k,du).$$

To this end, observe first that $F^*(i,j,t)$ can be decomposed as

$$F^*(i,j,t) = \hat{s}(i,j,t) + \sum_{k\neq j}\int_0^t F(k,j,t-u)\hat{s}(i,k,du). \tag{34}$$

Indeed, $\hat{s}(i,j,t)$ is the probability that the first transition takes place before time t and happens to be to state $j \in E$, and

$$\int_0^t F(k,j,t-u)\hat{s}(i,k,du) = \mathbb{P}^*\{x_v = j \text{ for some } v \leq t,\ x_{T_1} = k | x_0 = i\}.$$

In view of (29) and (34), Remark 3.1 yields

$$\begin{aligned} R^*(i,j,t) * h(j,t) &= R(j,j,t) * F^*(i,j,t) * h(j,t) \\ &= R(j,j,t) * \hat{s}(i,j,t) * h(j,t) + \sum_{k\neq j} F(k,j,t) * \hat{s}(i,k,t) * R(j,j,t) * h(j,t). \end{aligned}$$

Now recall from Proposition 6.3 in [15], for example, that

$$P_t(i,j) = \int_0^t h(j,t-s)R(i,j,ds).$$

Thus, by also using (27) we obtain

$$\begin{aligned} R^*(i,j,t) * h(j,t) &= \hat{s}(i,j,t) * P_t(j,j) + \sum_{k\neq j} R(j,j,t) * F(k,j,t) * \hat{s}(i,k,t) * h(j,t) \\ &= \hat{s}(i,j,t) * P_t(j,j) + \sum_{k\neq j} R(k,j,t) * \hat{s}(i,k,t) * h(j,t) \\ &= \sum_k \hat{s}(i,k,t) * P_t(k,j). \end{aligned}$$

This proves our assertion. □

3.2 The rate of convergence to equilibrium Now, our goal is to derive the rates of convergence of the mappings $t \mapsto s(i,t)$ to 0 and $t \mapsto R^*(i,j,t) * h(j,t)$ to ν_j, respectively. Due to (25) it is enough to analyze the case $i,j \neq 0$. To this end we shall first study the asymptotic behavior of the map $t \mapsto R^*(i,j,t)$. Since $R(j,j,\cdot)$ is a renewal function, we see from (29) and (33) that the asymptotics of $P^*_t(i,j)$ can be derived as an application of Theorem A.1 essentially if we can show that

$$F^*(i,j,t) * h(j,t) = o(\bar{F}(j,j,t)). \tag{35}$$

3.2.1 The tail structure of the travel times Let us first deal with the issue of finding the convergence rate of $\bar{F}(j,j,t) = 1 - F(j,j,t)$ to 0. To this end, we introduce the family of random variables

$$\Theta = \{(\theta^\ell_{i,j}), i,j \in E, \ell = 0,1,2,...\}, \tag{36}$$

such that any two random variables in Θ are independent, and for fixed pair (i,j) the random variables $\theta^k_{i,j}$ have $G(i,j,\cdot)$ as their common distribution function. To ease the notational complexity we assume that the law of $T_{n+1} - T_n$ only depends on ξ_n. We shall therefore drop the second sub-index from the elements of Θ. The random variables $(\theta_i^\ell)_{\ell\in\mathbb{N}}$ are independent copies of the sojourn time in state i. We shall prove Lemma 3.4 below under this additional assumption. The general case where $T_{n+1}-T_n$ depends both on ξ_n and ξ_{n+1} can be analyzed by similar means.

Let $N_k^{i,j}$ denote the number of times the embedded Markov chain $\{\xi_n\}_{n\in\mathbb{N}}$ visits state $k\in E$ before it visits state j, given $\xi_0=i$. By definition, $N_j^{i,j}=0$ with probability one. We denote by $\boldsymbol{N}^{i,j}$ the vector of length $|E|$ with entries $N_k^{i,j}$, and by $\boldsymbol{n}=(n_k)_{k\in E}$ an element of $\mathbb{N}^{|E|}$. Then we have

$$\bar F(i,j,t)=\mathbb{P}\left\{\theta_i^0+\sum_{k\neq j}\sum_{\ell=1}^{N_k^{i,j}}\theta_k^\ell>t\right\}. \tag{37}$$

With $G(k,t):=\mathbb{P}\{T_{n+1}-T_n\le t\big|\xi_n=k\}$, we can rewrite (37) as

$$\begin{aligned}\bar F(i,j,t) &= \sum_{\boldsymbol{n}}\mathbb{P}\left\{\theta_i^0+\sum_{k\neq j}\sum_{\ell=1}^{n_k}\theta_k^\ell>t\,\middle|\,\boldsymbol{N}^{i,j}=\boldsymbol{n}\right\}\mathbb{P}\{\boldsymbol{N}^{i,j}=\boldsymbol{n}\}\\ &= 1-G(i,t)*\sum_{\boldsymbol{n}}\mathop{*}_{k\neq j}G^{n_k}(k,t)\mathbb{P}\{\boldsymbol{N}^{i,j}=\boldsymbol{n}\}.\end{aligned} \tag{38}$$

Our goal is now to show that

$$\lim_{t\to\infty}\frac{\bar F(i,j,t)}{t^{-\alpha}L(t)}=\sum_{n\ge0}n\mathbb{P}\{N_0^{i,j}=n\}+\delta_{i,0} \tag{39}$$

for $j\neq0$. Here, $\delta_{i,0}=0$ if $i\neq0$ and $\delta_{i,0}=1$ otherwise. The first term on the right-hand-side, $\sum_{n\ge0}n\mathbb{P}\{N_0^{i,j}=n\}$, is the expected number of occurrences of state 0 under $\mathbb{P}$ before the first visit to state j, given $\xi_0=x_0=i$. This quantity is positive, due to Assumption 2.1. In order to prove (39), we need the following results which appear as Lemma 10 in [34].

Lemma 3.3 *Let $F_1,...,F_m$ be probability distribution functions such that, for all $j\neq i$, we have $\bar F_j(t)=o(\bar F_i(t))$. Then for any positive integers $n_1,\dots,n_m$,*

$$1-F_1^{n_1}*...*F_m^{n_m}(t)\sim n_i\bar F_i(t).$$

Moreover, for each $u>0$, there exists some $K_u<\infty$ such that

$$\frac{1-F_1^{n_1}*...*F_m^{n_m}(t)}{1-F_i^{n_i}(t)}\le K_u(1+u)^{n_i}$$

for all $t\ge0$.

We are now ready to prove (39).

Lemma 3.4 *Under the assumptions of Theorem 3.1 we have, for $j\neq0$,*

$$\lim_{t\to\infty}\frac{\bar F(i,j,t)}{t^{-\alpha}L(t)}=\sum_{n\ge0}n\mathbb{P}\{N_0^{i,j}=n\}+\delta_{i,0}>0.$$

Proof. Let us first prove that the expected number of occurrences of state 0 before the first return to state j occurs is finite. To this end, we put $p=\min\{p_{ij}:i,j\in E,i\neq j\}>0$. Since

$$\mathbb{P}\{N_0^{i,j}=n\}\le\mathbb{P}\{\xi_m\neq j\text{ for all }m\le n\}\le(1-p)^n,$$

we obtain

$$\sum_{n\ge0}n\mathbb{P}\{N_0^{i,j}=n\}\le\sum_{n\ge0}n(1-p)^n<\infty. \tag{40}$$

Now, we define a probability measure $\bar\mu$ on $\mathbb{N}^{|E|}$ by

$$\bar\mu\{\boldsymbol{n}\}=\mathbb{P}\{\boldsymbol{N}^{i,j}=\boldsymbol{n}\}$$

and put

$$A_{\boldsymbol{n}}(t)=G(i,t)\mathop{*}_{k\neq j}G^{n_k}(k,t).$$

Since $\frac{1-G(0,t)}{t^{-\alpha}L(t)} \to 1$ as $t \to \infty$, the first part of Lemma 3.3 yields

$$\lim_{t\to\infty} \frac{1-A_{\boldsymbol{n}}(t)}{t^{-\alpha}L(t)} = \lim_{t\to\infty} \left(\frac{1-G(i,t) \underset{k\neq j}{*} G^{n_k}(k,t)}{1-G(0,t)} \right) \left(\frac{1-G(0,t)}{t^{-\alpha}L(t)} \right) = n_0 + \delta_{i,0}.$$

From the definition of the measure $\bar{\mu}$, we obtain

$$\frac{1-\sum_{\boldsymbol{n}} A_{\boldsymbol{n}}(t)\mathbb{P}\{\boldsymbol{N}^{i,j}=\boldsymbol{n}\}}{t^{-\alpha}L(t)} = \mathbb{E}_{\bar{\mu}} \left\{ \frac{1-A_{\boldsymbol{n}}(t)}{t^{-\alpha}L(t)} \right\},$$

and so our assertion follows from the dominated convergence theorem if we can show that

$$\sup_t \frac{1-A_{\boldsymbol{n}}(t)}{t^{-\alpha}L(t)} \in L^1(\bar{\mu}). \tag{41}$$

To verify (41), we will use the second part of Lemma 3.3. For each $u>0$ there exists a constant K_u such that

$$\frac{1-A_{\boldsymbol{n}}(t)}{t^{-\alpha}L(t)} = \left(\frac{1-A_{\boldsymbol{n}}(t)}{1-G(0,t)} \right) \left(\frac{1-G(0,t)}{t^{-\alpha}L(t)} \right) \leq K_u(1+u)^{n_0+\delta_{i,0}} \sup_t \frac{1-G(0,t)}{t^{-\alpha}L(t)}.$$

Since

$$\lim_{t\to\infty} \frac{1-G(0,t)}{t^{-\alpha}L(t)} = 1,$$

and because we are only interested in the asymptotic behavior of the function $t \mapsto \frac{\bar{F}(i,j,t)}{t^{-\alpha}L(t)}$, we may with no loss of generality assume that

$$\sup_t \frac{1-G(0,t)}{t^{-\alpha}L(t)} = 1.$$

This yields

$$\sup_t \frac{1-A_{\boldsymbol{n}}(t)}{t^{-\alpha}L(t)} \leq K_u(1+u)^{n_0+\delta_{i,0}}. \tag{42}$$

From (40) and (42) we get

$$\mathbb{E}_{\bar{\mu}} \left\{ \sup_t \frac{1-A_{\boldsymbol{n}}(t)}{t^{-\alpha}L(t)} \right\} \leq K_u(1+u)^{\delta_{i,0}} \sum_{k=0}^{\infty} (1-p)^k (1+u)^k.$$

Choosing $u < \frac{p}{1-p}$ we obtain $\beta := (1-p)(1+u) < 1$ and so the assertion follows from

$$\mathbb{E}_{\bar{\mu}} \left\{ \sup_t \frac{1-A_{\boldsymbol{n}}(t)}{t^{-\alpha}L(t)} \right\} \leq K_u(1+u)^{\delta_{i,0}} \sum_{n\geq 0} \beta^n < \infty.$$

□

So far, we have shown that $\bar{F}(i,j,t) \sim t^{-\alpha}L(t) \left(\sum_{n\geq 0} n\mathbb{P}\{N_0^{i,j}=n\} + \delta_{i,0} \right)$ for $j \neq 0$. In view of Lemmas 3.3 and 3.4, the representation (34) of $F^*(i,j,t)$ yields a similar result for the stationary setting.

Corollary 3.1 *For all $i,j \neq 0$ we have*

$$\lim_{t\to\infty} \frac{\bar{F}^*(i,j,t)}{t^{-\alpha}L(t)} = \sum_{n\geq 0} n\mathbb{P}\{N_0^{i,j}=n\}.$$

Proof. Due to (34), (30) and (14) we can write

$$F^*(i,j,t) = \frac{p_{ij}}{m_{i,j}} \int_0^t (1-G(i,j,s))ds + \sum_{k\neq j} \frac{p_{ik}}{m_{i,k}} \int_0^t F(k,j,t-u)(1-G(i,k,u))du, \tag{43}$$

and therefore

$$\bar{F}^*(i,j,t) = \frac{p_{ij}}{m_{i,j}} \int_t^\infty (1-G(i,j,s))ds + \sum_{k\neq j} p_{ik} \left(1 - \frac{1}{m_{i,k}} \int_0^t F(k,j,t-u)(1-G(i,k,u))du \right). \tag{44}$$

We will now show that

$$\lim_{t\to\infty} \frac{\int_t^\infty (1-G(i,j,s))ds}{t^{-\alpha}L(t)} = 0 \quad \text{if} \quad p_{ij} > 0 \quad \text{and} \quad i \neq 0. \tag{45}$$

To this end, we first apply Proposition 1.5.10 in [8]: if g is a function on $\mathbb{R}_+$ that satisfies $g(t) \sim t^{-\beta}L(t)$ for $\beta > 1$, then

$$\int_t^\infty g(s)ds \sim \frac{t^{-\beta+1}}{\beta-1}L(t).$$

Together with Assumption 2.2 (iii) this proposition implies that

$$\lim_{t\to\infty} \frac{\int_t^\infty h(i,s)ds}{t^{-\alpha}L(t)} = 0, \quad i \neq 0, \tag{46}$$

where $h(i,\cdot)$ is the tail of the distribution of the sojourn time in state i, defined in (12). The representation

$$h(i,t) = 1 - \sum_{j\in E} p_{ij}G(i,j,t) = \sum_{j\in E} p_{ij}(1-G(i,j,t)), \tag{47}$$

along with (46) implies (45).

In order to find the decay rate of the remaining terms of (44), recall first that $N_k^{i,j}$ is the number of visits to state k before reaching state j, given $\xi_0 = i$, and not counting the first one in the case $k = i$.

$$\mathbb{P}\{N_0^{i,j} = n\} = \sum_{k\notin\{0,j\}} p_{ik}\mathbb{P}\{N_0^{k,j} = n\} + p_{i0}\mathbb{P}\{N_0^{i,0} = n-1\},$$

and so

$$\sum_{n\geq 0} n\mathbb{P}\{N_0^{i,j} = n\} = \sum_{k\neq j} p_{ik} \sum_{n\geq 0} n\mathbb{P}\{N_0^{k,j} = n\} + p_{i0} \sum_{n\geq 0} \mathbb{P}\{N_0^{0,j} = n\}. \tag{48}$$

Note that Assumption 2.1 implies $\sum_{n\geq 0} \mathbb{P}\{N_0^{0,j} = n\} = 1$. Since $\frac{1}{m_{i,j}} \int_0^t (1-G(i,j,s))ds$ is a distribution function whose tail is $\frac{1}{m_{i,j}} \int_t^\infty (1-G(i,j,s))ds$, Lemmas 3.3, 3.4 together with equations (45) and (48) imply that

$$\begin{aligned}
\lim_{t\to\infty} \frac{\sum_{k\neq j} p_{ik}\left(1 - \frac{1}{m_{i,k}}\int_0^t F(k,j,t-u)(1-G(i,k,u))du\right)}{t^{-\alpha}L(t)} &= \sum_{k\neq j} p_{ik} \sum_{n\geq 0} n\mathbb{P}\{N_0^{k,j} = n\} + p_{i0} \\
&= \sum_{n\geq 0} n\mathbb{P}\{N_0^{i,j} = n\}.
\end{aligned}$$

This completes the proof.

□

The next result shows that the first term on the right-hand-side of (33) converges to zero sufficiently fast.

COROLLARY 3.2 *For all $i \neq 0$ we have*

$$\lim_{t\to\infty} \frac{s(i,t)}{t^{-\alpha+1}L(t)} = 0.$$

PROOF. The proof is an immediate consequence of Corollary 3.1 since $\bar{F}^*(i,j,t) \geq s(i,t)/\nu_i$. □

3.2.2 The tail structure of $R^* * h$ So far, we have analyzed the tail structure of the distribution of the travel time between states i and j $(i,j \neq 0)$ in the stationary regime. We are now going to study the tail structure of $R^*(i,j,t) * h(j,t)$.

LEMMA 3.5 *There exists $C_j > 0$ such that, for $j \neq 0$,*

$$\lim_{t\to\infty} \frac{R^*(i,j,t) * h(j,t) - \nu_j}{t^{-\alpha+1}L(t)} = \frac{C_j}{\alpha-1}$$

for all $i \in E$, $i \neq 0$.

Proof. Let us fix $i, j \in E$, $i, j \neq 0$. Using the representation (29) for the function $R^*(i, j, \cdot)$, we need to show that

$$\lim_{t\to\infty} \frac{R(j,j,t) * F^*(i,j,t) * h(j,t) - \nu_j}{t^{-\alpha+1}L(t)} = \frac{C_j}{\alpha - 1}.$$

It follows from Fubini's theorem that

$$\begin{aligned} \int_0^\infty F^*(i,j,t) * h(j,t)dt &= \int_0^\infty \int_0^\infty h(j,t-s)\mathbf{1}_{\{s\leq t\}}F^*(i,j,ds)\,dt \\ &= \int_0^\infty \int_s^\infty h(j,t-s)dt\,F^*(i,j,ds) \\ &= \int_0^\infty h(j,t)dt \\ &= m_j, \end{aligned}$$

where m_j is the mean sojourn time at state $j \in E$ as defined in (9). Suppose now that we can show that the continuous non-negative function z defined by

$$z(i,j,t) := F^*(i,j,t) * h(j,t) \tag{49}$$

satisfies $z(i,j,t) = o(\bar{F}(i,j,t))$, is directly Riemann integrable[3] and of bounded variation, and its total variation function $z^*(i,j,\cdot)$, defined in (57) below, satisfies $z^*(i,j,t) = O(t^{-\alpha+1}\hat{L}(t))$, where we define

$$\hat{L}(t) = L(t)\sum_{n\geq 0} n\mathbb{P}\{N_0^{j,j} = n\}.$$

Then Theorem A.1 yields

$$\frac{m_j}{\eta_j} - \int_0^t z(i,j,t-s)R(j,j,ds) \quad \sim \quad -\frac{m_j}{(\alpha-1)\eta_j^2}t^{-\alpha+1}\hat{L}(t), \tag{50}$$

because $R(j,j,t) = \sum_{n\geq 0} F^n(j,j,t)$ is a renewal function, $F(j,j,\cdot)$ is nonsingular and $\bar{F}(j,j,t) \sim t^{-\alpha}\hat{L}(t)$ for $j \neq 0$. By Propositions 5.5 and 6.12 in [15], $\nu_j = \frac{m_j}{\eta_j}$, and so (50) would yield

$$\lim_{t\to\infty} \frac{R(j,j,t) * F^*(i,j,t) * h(j,t) - \nu_j}{t^{-\alpha+1}L(t)} = \frac{m_j}{(\alpha-1)\eta_j^2}\sum_{n\geq 0} n\mathbb{P}\{N_0^{j,j} = n\}$$

and hence prove our assertion with

$$C_j := \frac{m_j}{\eta_j^2}\sum_{n\geq 0} n\mathbb{P}\{N_0^{j,j} = n\}. \tag{51}$$

We will now establish that *(i)* $z(i,j,t) = o(\bar{F}(i,j,t))$, that *(ii)* $z^*(i,j,t) = O(t^{-\alpha+1}\hat{L}(t))$ and $z(i,j,\cdot)$ is of bounded variation, and that *(iii)* $z(i,j,\cdot)$ is directly-Riemann integrable.

(i) Since $i, j \neq 0$ and because $p_{i,0} > 0$ the probability that the semi-Markov process x visits the state 0 before it reaches the state j is positive. Thus, it follows from Assumption 2.2 (iii) and from Corollary 3.1 that

$$h(j,t) = o(\bar{F}^*(i,j,t)). \tag{52}$$

Now it follows from (52) and Lemma 3.3 that

$$\bar{F}^*(i,j,t) + z(i,j,t) = 1 - F^*(i,j,t) * (1 - h(j,t)) \sim \bar{F}^*(i,j,t). \tag{53}$$

Hence $z(i,j,t) = o(\bar{F}^*(i,j,t))$, and so $z(i,j,t) = o(\bar{F}(i,j,t))$ by Lemma 3.4 and Corollary 3.1.

(ii) Assumption 2.2 (iv) along with (38), the representation (43) of $F^*(i,j,\cdot)$ and (47) guarantees that $F^*(i,j,\cdot)$ and $h(j,\cdot)$ have a bounded continuous densities $f^*(i,j,\cdot)$ and $h'(j,\cdot)$, respectively. As a result, $z(i,j,\cdot)$ is absolutely continuous with density

$$z'(i,j,t) = f^*(i,j,t) + \int_0^t h'(j,t-s)f^*(i,j,s)ds. \tag{54}$$

[3] See Appendix B for the definition of directly Riemann integrability.

Since $h'(j,t-s) \le 0$,

$$|z'(i,j,t)| \le f^*(i,j,t) - \int_0^t h'(j,t-s) f^*(i,j,s) ds. \tag{55}$$

In terms of the distribution function $g(j,t) := 1 - h(j,t)$ we have

$$\begin{aligned} -\int_0^t h'(j,t-s) f^*(i,j,s) ds &= -\frac{\partial}{\partial t}\int_0^t h(j,t-s) f^*(i,j,s) ds + f^*(i,j,t) \\ &= \frac{\partial}{\partial t}\left(g(j,t) * F^*(i,j,t)\right). \end{aligned} \tag{56}$$

As a result the total variation function $z^*(i,j,\cdot)$ satisfies

$$z^*(i,j,t) = \int_t^\infty |z'(i,j,s)| ds \le \bar{F}^*(i,j,t) + \left[1 - g(j,t) * F^*(i,j,t)\right]. \tag{57}$$

Thus, $z^*(i,j,t) \le 2$ and $z(i,j,\cdot)$ is of bounded variation. Lemma 3.3 and (52) together with (57) give $z^*(i,j,t) = O(\bar{F}^*(i,j,t))$, and so Corollary 3.1 yields $z^*(i,j,t) = O(t^{-\alpha+1}\hat{L}(t))$.

(iii) In order to prove that z is directly Riemann integrable note first that since $h \ge 0$ is decreasing and by Assumption 2.2 (i) it is integrable, therefore by Lemma B.1 (i) it is directly Riemann integrable. Now Lemma B.1 (ii) implies that z is directly Riemann integrable.

□

3.2.3 Proof of the central limit theorem for semi-Markov processes We are now ready to prove the main result of this section.

Proof of Theorem 3.1: By (33) we have the representation

$$P_t^*(i,j) = \frac{s(i,t)}{\nu_i}\delta_{ij} + \int_0^t h(j,t-s) R^*(i,j,ds)$$

for the conditional probability that $x_t = j$, given $x_0 = i$. Due to Corollary 3.2 and Lemma 3.5,

$$\lim_{t\to\infty} \frac{P_t^*(i,j) - \nu_j}{t^{-\alpha+1}L(t)} = \frac{C_j}{\alpha - 1} \qquad (i,j \ne 0),$$

where C_j is given by (51). With $H = \frac{3-\alpha}{2}$ it follows from (25) that

$$\lim_{t\to\infty} \frac{\gamma(t)}{t^{2H-2}L(t)} = \frac{1}{(2-2H)} \sum_{i,j\in E} ij\nu_i C_j. \tag{58}$$

By Lemma 3.1 this proves the existence of a constant c such that the finite dimensional distributions of the processes $\left(\frac{1}{\varepsilon^{1-H}\sqrt{L(\varepsilon^{-1})}} X_t^\varepsilon\right)_{0\le t<\infty}$ converge weakly to the finite dimensional distributions of the fractional Brownian motion process cB^H as $\varepsilon \downarrow 0$, and c is given by

$$c^2 = \frac{1}{2H(1-H)(2H-1)} \sum_{i,j\in E} ij\nu_i \frac{m_j}{\eta_j^2} \sum_{n\ge 0} n\mathbb{P}\{N_0^{j,j} = n\} = \frac{1}{2H(1-H)(2H-1)}\mu \sum_{j\in E} j\frac{m_j}{\eta_j^2} \sum_{n\ge 0} n\mathbb{P}\{N_0^{j,j} = n\}.$$

In order to establish tightness, we proceed in two steps.

(i) We first establish the existence of a constant $C < \infty$ such that

$$\mathrm{Var}(t) \le Ct^{2H}L(t) \qquad \text{for all} \qquad t \ge 0. \tag{59}$$

In view of (22), we can choose a sufficiently large $T \in \mathbb{N}$ that satisfies

$$\mathrm{Var}(t) \le 2c^2 t^{2H} L(t) \qquad \text{for all} \qquad t \ge T.$$

In terms of the random variables $Y^i := \int_{(i-1)t/T}^{it/T} Y_s ds$ $(i = 1, 2, \ldots, T)$ we have

$$\mathrm{Var}\left(\int_0^t Y_s ds\right) = \mathrm{Var}\left(\sum_{i=1}^T \int_{(i-1)t/T}^{it/T} Y_s ds\right) = \sum_{i,j=1}^T \mathrm{Cov}(Y^i, Y^j).$$

Since the mean zero Gaussian process $(\int_0^\cdot Y_s ds)$ has stationary increments, the sequence $Y^1, Y^2, \dots, Y^T$ is stationary. Thus, the Hölder inequality yields

$$\mathrm{Cov}(Y^i, Y^j) \le \sqrt{\mathrm{Var}(Y^i)}\sqrt{\mathrm{Var}(Y^j)} = \mathrm{Var}(Y^0),$$

and so

$$\mathrm{Var}(t) \le T^2 \mathrm{Var}\left(\frac{t}{T}\right) \quad \text{for} \quad 0 \le t \le T.$$

In view of (25), the function γ is bounded: $\|\gamma\|_\infty < \infty$. Hence for $s \in [0,1]$ the representation (20) shows that

$$\mathrm{Var}(s) = 2\int_0^s \int_0^v \gamma(u)dudv \le s^2\|\gamma\|_\infty \le s^{2H}\|\gamma\|_\infty \quad \text{because} \quad H \in \left(\frac{1}{2}, 1\right).$$

This yields (59) with $C := \max\{2c^2, T^{2-2H}\|\gamma\|_\infty\}$ (after putting $L \equiv 1$ on $[0,T]$).

(ii) Let us now denote by $Z^\varepsilon = (Z_t^\varepsilon)$ the mean zero Gaussian process with stationary increments defined by

$$Z_t^\varepsilon := \frac{1}{\varepsilon^{1-H}\sqrt{L(\varepsilon^{-1})}} X_t^\varepsilon \qquad (0 \le t \le T).$$

Due to Theorem 12.3 in [7], the family of stochastic processes (Z^ε) is tight if the following moment condition is satisfied for some constants $\delta > 1$ and $\hat{C} < \infty$ and for all sufficiently small ε:

$$\mathbb{E}[Z_{t_2}^\varepsilon - Z_{t_1}^\varepsilon]^2 \le \hat{C}|t_2 - t_1|^\delta \quad \text{for all} \quad 0 \le t_1 \le t_2 \le T. \tag{60}$$

In view of (21), we have

$$\mathbb{E}[Z_{t_2}^\varepsilon - Z_{t_1}^\varepsilon]^2 = \varepsilon^2 \mathrm{Var}\left(\frac{t_2 - t_1}{\varepsilon}\right)\frac{1}{\varepsilon^{2-2H}L(\varepsilon^{-1})},$$

and so it follows from step (i) that

$$\mathbb{E}[Z_{t_2}^\varepsilon - Z_{t_1}^\varepsilon]^2 \le C(t_2 - t_1)^{2H}\frac{L\left(\varepsilon^{-1}(t_2 - t_1)\right)}{L(\varepsilon^{-1})}. \tag{61}$$

Since L is slowly varying, $\frac{L(\varepsilon^{-1}u)}{L(\varepsilon^{-1})}$ tends to 1 as $\varepsilon \to 0$, and this convergence is uniform in u over compact sets ([8], Theorem 1.2.1). Thus, there exists $\varepsilon^* > 0$ such that

$$\frac{L(\varepsilon^{-1}u)}{L(\varepsilon^{-1})} \le 2 \quad \text{for all} \quad 0 \le u \le T$$

if $\varepsilon < \varepsilon^*$, and so (60) follows from (61) with $\hat{C} := 2C$. □

4. Approximating Integrals with respect to fractional Brownian motion In this section we prove an approximation result for stochastic integrals which contains Theorem 2.1 as a special case. More precisely we give conditions which guarantee that for a sequence of processes $\{(\Psi^n, Z^n)\}_{n\in\mathbb{N}}$ the convergence $\mathcal{L}\text{-}\lim_{n\to\infty}(\Psi^n, Z^n) = (\Psi, B^H)$ implies the convergence $\mathcal{L}\text{-}\lim_{n\to\infty}\left(\Psi^n, Z^n, \int \Psi^n dZ^n\right) = \left(\Psi, B^H, \int \Psi dB^H\right)$. We follow the notational convention in [38] that $\int XdY$ (without displaying the running variable) denotes $\int X_{s-}dY_s$.

All stochastic integrals in this section are understood as limits in probability of Stieltjes-type sums which can be described as follows: given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a filtration $\{\mathcal{F}_t : t \ge 0\}$ of sub-sigma-fields of $\mathcal{F}$ consider two adapted stochastic processes ϕ and Z; we say that the integral $\int \phi dZ$ exists if for any $T < \infty$ and for each sequence of partitions $\{\tau^l\}_{l\in\mathbb{N}}$, $\tau^l = (\tau_1^l, \tau_2^l, \dots, \tau_{N_l}^l)$, of the interval $[0,T]$ that satisfies $\lim_{l\to\infty}\max_i|\tau_{i+1}^l - \tau_i^l| = 0$,

$$\int_0^T \phi_{s-}dZ_s = \mathbb{P}\text{-}\lim_{l\to\infty}\sum_i \phi_{\tau_i^l}(Z_{\tau_{i+1}^l} - Z_{\tau_i^l}), \tag{62}$$

where $\mathbb{P}$-lim denotes the limit in probability. This definition of stochastic integrals applies to the usual semimartingale setting where ϕ is a process in $\mathbb{D}$ and where Z is a semimartingale. If $Z = B^H$ is a fractional Brownian motion process with Hurst coefficient $H > \frac{1}{2}$, the limit in (62) exists for a large class of integrands, including continuous semimartingales and C^1-functions of fractional Brownian motion. In particular, the stochastic integral $\int B^H dB^H$ exists in the sense of (62), and for the continuous semimartingale Ψ we have the following integration by parts formula, due to [39]:

$$-\int B^H d\Psi + \Psi B^H = \int \Psi dB^H. \tag{63}$$

Before we state the main result of this section, we recall that a sequence $\{\Psi^n\}_{n\in\mathbb{N}}$ of semimartingales defined on probability spaces $(\Omega^n, \mathcal{F}^n, \mathbb{P}^n)$ is called *good* in the sense of [22] if, for any sequence $\{Z^n\}_{n\in\mathbb{N}}$ of càdlàg adapted processes, the convergence $\mathcal{L}\text{-}\lim_{n\to\infty}(\Psi^n, Z^n) = (\Psi, Z)$ implies the convergence

$$\mathcal{L}\text{-}\lim_{n\to\infty}\left(\Psi^n, Z^n, \int Z^n d\Psi^n\right) = \left(\Psi, Z, \int Z d\Psi\right).$$

Remark 4.1 *Any semi-martingale Ψ can be written as $\Psi = M + A$, in which M is a local martingale with $M_0 = 0$ and A is an adapted finite variation process. We will denote by $[M,M]$ the quadratic variation of M and by $|A|$ the total variation of A. A sequence $(\Psi^n\}_{n\in\mathbb{N}}$ ($\Psi^n = M^n + A^n$) of semi-martingales is good on $[0,T]$ if the sequences $(\mathbb{E}_n\{[M^n,M^n]_T\})_{n\in\mathbb{N}}$ and $(\mathbb{E}_n\{|A|_T\})_{n\in\mathbb{N}}$ are bounded; see [22, Theorem 4.1] for details.*

In the following we will denote by $\{\Psi^n\}_{n\in\mathbb{N}}$ a sequence of "good" semimartingales and by $\{Z^n\}_{n\in\mathbb{N}}$ a sequence of $\mathbb{D}$-valued stochastic processes defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and assume that the following conditions are satisfied.

Assumption 4.1 *(i) The sample paths of the processes Z^n are almost surely of zero quadratic variation on compact sets, and $\mathbb{P}\{Z_0^n = 0\} = 1$.*

(ii) The stochastic integrals $\int \Psi^n dZ^n$ and $\int Z^n dZ^n$ exist in the sense of (62), and the sample paths $t \mapsto \int_0^t Z_{s-}^n dZ_s^n$ and $t \mapsto \int_0^t \Psi_{s-}^n dZ_s^n$ are càdlàg.

We are now ready to state the main theorem of this section. Its proof requires some preparation which will be carried out below.

Theorem 4.1 *Let $\{\Psi^n\}_{n\in\mathbb{N}}$ be a sequence of good semimartingales and let $\{Z^n\}_{n\in\mathbb{N}}$ be a sequence of $\mathbb{D}$-valued stochastic processes that satisfy Assumption 4.1. If Ψ is a continuous semimartingale and if B^H is a fractional Brownian motion process with Hurst parameter $H > \frac{1}{2}$, then the convergence $\mathcal{L}\text{-}\lim_{n\to\infty}(\Psi^n, Z^n) = (\Psi, B^H)$ implies the convergence*

$$\mathcal{L}\text{-}\lim_{n\to\infty}\left(\Psi^n, Z^n, \int \Psi^n dZ^n\right) = \left(\Psi, B^H, \int \Psi dB^H\right).$$

Before we turn to the proof of Theorem 4.1, we consider an example where Assumption 4.1 can indeed be verified.

Example 4.1 *Let $\{H_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers with $H_n > \frac{1}{2}$, and assume that $\lim_{n\to\infty} H_n = H > \frac{1}{2}$. Let Z^n be a fractional Brownian motion process with Hurst parameter H_n and let Ψ be a continuous semimartingale independent of Z^n for all n. Since $H_n > \frac{1}{2}$, the processes Z^n have zero quadratic variation. Moreover, $\mathcal{L}\text{-}\lim_{n\to\infty} Z^n = B^H$ because the centered Gaussian processes Z^n and B^H are uniquely determined by their covariation functions and all stochastic integrals exists in the sense of (62). Thus, Theorem 4.1 yields*

$$\mathcal{L}\text{-}\lim_{n\to\infty}\left(Z^n, \int \Psi dZ^n\right) = \left(B^H, \int \Psi dB^H\right).$$

We prepare the proof of Theorem 4.1 with the following simple lemma.

LEMMA 4.1 *Under the assumptions of Theorem 4.1 the processes $[Z^n]$ and $[Z^n, \Psi^n]$ defined by*

$$[Z^n]_t := (Z^n_t)^2 - 2\int_0^t Z^n_{s-} dZ^n_s \quad \text{and} \quad [Z^n, \Psi^n]_t := Z^n_t \Psi^n_t - \int_0^t Z^n_{s-} d\Psi^n_s - \int_0^t \Psi^n_{s-} dZ^n_s,$$

have $\mathbb{P}$-a.s. sample paths which are equal to zero.

PROOF. It follow from the representation of the stochastic integrals $\int Z^n dZ^n$, $\int \Psi^n dZ^n$ and $\int Z^n d\Psi^n$ as probabilistic limits of Stieltjes-type sums that, for any t and each sequence of partitions $\{\tau^l\}_{l\in\mathbb{N}}$ of $[0,t]$ with $\lim_{l\to\infty} \max_i |\tau^l_{i+1} - \tau^l_i|$,

$$[Z^n] = \mathbb{P}\text{-}\lim_{l\to\infty} \sum_i (Z^n_{\tau^l_{i+1}} - Z^n_{\tau^l_i})^2 \text{ and } [Z^n, \Psi^n] = \mathbb{P}\text{-}\lim_{l\to\infty} \sum_i (Z^n_{\tau^l_{i+1}} - Z^n_{\tau^l_i})(\Psi^n_{\tau^l_{i+1}} - \Psi^n_{\tau^l_i}).$$

Since a typical sample path of the stochastic integrals $\int \Psi^n dZ^n$ and $\int Z^n d\Psi^n$ is in $\mathbb{D}$, we can apply the same arguments as in the proof of Theorem II.6.25 in [48] in order to obtain the inequality $\mathbb{P}\left\{[Z^n, \Psi^n]^2_t \le [Z^n]_t [\Psi^n]_t\right\} = 1$. Thus, our assertion follows from $\mathbb{P}\{[Z^n]_t = 0 \text{ for all } t \ge 0\} = 1$. □

For the proof of Theorem 4.1 we will also need the following result.

LEMMA 4.2 *(i) Let $\mathbb{C}$ be the space of all real valued continuous functions. For $n \in \mathbb{N}$, let $\alpha_n, \beta_n \in \mathbb{D}$ and assume that the sequence $\{(\alpha_n, \beta_n)\}_{n\in\mathbb{N}}$ converges in the Skorohod topology to $(\alpha, \beta) \in \mathbb{C}\times\mathbb{C}$. Then, on compact intervals, the process*

$$\gamma_n = (\gamma_n(t))_{t\ge 0} \quad \text{defined by} \quad \gamma_n(t) = \alpha_n(t)\beta_n(t)$$

converges to $\alpha\beta = (\alpha(t)\beta(t))_{t\ge 0}$ in the Skorohod topology on $\mathbb{D}$.

(ii) Let $\{(Y^n, Z^n)\}_{n\in\mathbb{N}}$ be a sequence of $\mathbb{D}$-valued random variables defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ that converges in law to (Y, Z). If $\mathbb{P}\{(Y, Z) \in \mathbb{C}\times\mathbb{C}\} = 1$, then

$$\mathcal{L}\text{-}\lim_{n\to\infty} \{(Y^n_t Z^n_t)_{0\le t\le T}\} = (Y_t Z_t)_{0\le t\le T}$$

holds for all $T < \infty$.

PROOF. Since α and β are continuous, (i) follows from Lemma 2.1 in [38]. The second assertion follows from (i) and Skorohod's representation theorem. □

We are now ready to finish the proof of Theorem 4.1.

PROOF OF THEOREM 4.1: Since $\{\Psi^n\}_{n\in\mathbb{N}}$ is a sequence of good semimartingales and because a typical sample path of a fractional Brownian motion process is continuous, we deduce from Theorem 2.2 in [38] and from Lemma 4.2 (ii) that

$$\mathcal{L}\text{-}\lim_{n\to\infty} \left(\Psi^n, Z^n, \int Z^n d\Psi^n\right) = \left(\Psi, B^H, \int B^H d\Psi\right) \text{ and } \mathcal{L}\text{-}\lim_{n\to\infty} (\Psi^n Z^n) = \left(\Psi B^H\right), \tag{64}$$

respectively. By the continuous mapping theorem, it follows from (64), from Lemma 4.1 and from the integration by parts formula for fractional Brownian motion (63) that the finite dimensional distributions of the processes

$$\left(\Psi^n, Z^n, \int \Psi^n dZ^n\right) = \left(\Psi^n, Z^n, -\int Z^n d\Psi^n + \Psi^n Z^n\right)$$

converge weakly to the finite dimensional distributions of the process

$$\left(\Psi, B^H, -\int B^H d\Psi + \Psi B^H\right) = \left(\Psi, B^H, \int \Psi dB^H\right).$$

It also follows from (64) that the sequence $\left\{\int Z^n d\Psi^n\right\}_{n\in\mathbb{N}}$ is C-tight and the sequence $\{\Psi^n Z^n\}_{n\in\mathbb{N}}$ is tight. By Corollary VI.3.33 in [33] the sum of a tight sequence of stochastic processes with a sequence of C-tight processes is tight. Thus, continuity of the processes ΨB^H and $\int B^H d\Psi$ yields tightness of the sequence $\left\{\int \Psi^n dZ^n\right\}_{n\in\mathbb{N}}$. This shows that

$$\mathcal{L}\text{-}\lim_{n\to\infty} \left(\Psi^n, Z^n, \int \Psi^n dZ^n\right) = \left(\Psi, B^H, \int \Psi dB^H\right).$$

□

In view of Lemma 4.1 and the integration by parts formula for fractional Brownian motion processes we also have the following approximation result for the integral of fractional Brownian motion process with respect to itself.

PROPOSITION 4.1 *Under the assumption of Theorem 4.1 it holds that*

$$\mathcal{L}\text{-}\lim_{n\to\infty}\int_0^t Z^n_{s-}dZ^n_s = \int_0^t B^H_s dB^H_s.$$

PROOF. By Lemma 4.1

$$(Z^n_t)^2 = 2\int_0^t Z^n_{s-}dZ^n_s \qquad \mathbb{P}\text{-a.s.}$$

Thus, in view of Lemma 4.2 (ii) and the Itô formula for fractional Brownian motion the sequence $\{(Z^n_t)^2\}_{n\in\mathbb{N}}$ converges in distribution to

$$(B^H_t)^2 = 2\int_0^t B^H_s dB^H_s.$$

This yields the assertion. □

We finish this section with the proof of Theorem 2.1.

PROOF OF THEOREM 2.1: In terms of the processes Y and X^ε introduced in (19) and (21), respectively, we have

$$\begin{aligned}\mathcal{L}\text{-}\lim_{N\to\infty}\frac{1}{\sqrt{N}}(X^{\varepsilon,N}_t)_{0\le t\le T} &= \mathcal{L}\text{-}\lim_{N\to\infty}\left(\int_0^t\left(\frac{1}{\sqrt{N}}\sum_{a\in\mathbb{A}}\Psi_s x^a_{s/\varepsilon}-\sqrt{N}\mu\Psi_s\right)ds\right)_{0\le t\le T}\\ &= \left(\int_0^t \Psi_s Y_{s/\varepsilon}ds\right)_{0\le t\le T}\\ &= \left(\int_0^t \Psi_s dX^\varepsilon_s\right)_{0\le t\le T}\end{aligned}$$

and

$$\int_0^t X^\varepsilon_s dX^\varepsilon_s = \int_0^t X^\varepsilon_s Y_{s/\varepsilon}\,ds$$

where all the stochastic integrals have continuous sample paths. By Theorem 3.1,

$$\mathcal{L}\text{-}\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon^{1-H}\sqrt{L(\varepsilon^{-1})}}X^\varepsilon = c\,B^H$$

with $H > \frac{1}{2}$ and the sequence of semi-martingales consisting of a single element Ψ is good as a result of Remark 4.1 since by assumption $E\{[\Psi,\Psi]_T\} < \infty$ and $\mathbb{E}\{|A|_T\} < \infty$. Therefore, the assertion follows from Theorem 4.1 if we can show that the processes X^ε have zero quadratic variation on compact time intervals and that the stochastic integrals $\int \Psi dX^\varepsilon$ and $\int X^\varepsilon\, dX^\varepsilon$ exist as the probabilistic limits of Stieltjes-type sums. These properties, however, follow from (21) by direct computation. □

5. Conclusion & Discussion We proved a functional central limit theorem for stationary semi-Markov processes and an approximation result for stochastic integrals of fractional Brownian motion. Our motivation was to provide a mathematical framework for analyzing microstructure models where stock prices are driven by the demand of many small investors. We identified inertia as a possible source of long range dependencies in financial time series and showed how investor inertia can lead to stock price models driven by a fractional Brownian motion. Several avenues are open for further research.

In our financial market model, investors do not react to changes in the stock price. Over short time periods, such an assumption might be justified for small, non-professional investors, but incorporating feedback effects, whereby traders' investment decisions can be influenced by asset prices, would be an important next step from an economic point of view. This, however, leads to a significant increase in the complexity of the dynamics, as discussed in Remark 2.4.

Financial market models where stock prices are driven by the demand of many heterogenous agents constitute another mathematical challenge. Heterogeneity among traders has been identified as a key component affecting the dynamics of stock prices, and in recent years an extensive literature on the mathematical modelling of heterogeneity and interactions in financial markets has appeared. But the analysis is usually confined to models of active market participants trading every period, and it is therefore of interest to introduce heterogeneity and interaction into our model. One approach would be to build on the case study in Section 2.3 to look at models where inert traders (who alone would lead to a limit fractional Brownian motion price process) interact with active traders (who alone would lead to a limit standard Brownian motion price process), and to include mechanisms by which their *interaction* leads to efficient markets. One could also try incorporate *strategically* interacting institutional investors; see [4] for a possible game theoretic framework.

For long-term economic models, it is important to look at time-varying rates of long-range dependence, where the variation is caused by global economic factors, or regime changes. At some times, the market may be efficient, for example in a bullish exuberant economy like the late 1990s (see Figure 1), while at other times, the Joseph effect may be prominent, such as during a recession or period of economic nervousness as in the early 1990s. The mathematical challenge is then to adapt the limit theorem of this paper to stochastically varying measures of inertia.

Appendix A. The key renewal theorem in the heavy tailed case In this appendix we recall a result of Heath et al. ([31]) on the rate of convergence in the key renewal theorem in the heavy tailed case.

THEOREM A.1 *Let F be a distribution with domain $[0,\infty)$ satisfying*

$$\bar{F}(t) = 1 - F(t) \sim t^{-\alpha}\hat{L}(t)$$

for some $1 < \alpha < 2$, and where $\hat{L}$ is a slowly varying function at infinity. Assume that F^n is nonsingular for some $n \geq 1$. Let $\kappa = \int_0^\infty \bar{F}(x)dx$ be the expected value and denote by U the renewal function associated with F, that is,

$$U = \sum_{n=0}^{\infty} F^n.$$

Let z be a continuous, non-negative function of bounded variation on $[0,\infty)$, such that $\lim_{t\to\infty} z(t) = 0$. That is, $z(t) = \int_t^\infty \zeta(dy)$ for some finite signed measure ζ on $[0,\infty)$. Let z^* denote the total variation function of ζ. That is,

$$z^*(t) = \int_t^\infty |\zeta|(dy).$$

We will also assume that z is a directly Riemann integrable function *(see [16], p.295 for a definition) on $[0,\infty)$, such that $z(t) = o(\bar{F}(t))$and that*

$$z^*(t) = O\left(t^{-\alpha+1}\hat{L}(t)\right). \tag{65}$$

Let $\lambda = \int_0^\infty z(t)dt < \infty$. Then the function $h : \mathbb{R}_+ \to \mathbb{R}_+$ defined by

$$h(t) = \frac{\lambda}{\kappa} - \int_0^t z(t-s)U(ds)$$

satisfies

$$h(t) \sim -\frac{\lambda}{(\alpha-1)\kappa^2} t^{-\alpha+1}\hat{L}(t).$$

PROOF. The proof follows from the Remark on page 11 of [31]. □

Appendix B. Directly Riemann Integrable Functions The proof of our approximation result uses the notion of directly Riemann integrability. A real valued function g on $\mathbb{R}_+$ is called directly Riemann integrable if

$$\lim_{a\to 0+} \sum_{n=1}^{\infty} a \inf_{(n-1)a \leq t \leq na} g(t) = \lim_{a\to 0+} \sum_{n=1}^{\infty} a \sup_{(n-1)a \leq t \leq na} g(t) = \int g(t)dt.$$

These sums would be Riemann sums if not for the infinite limit of summation. If a function is directly Riemann integrable, it is Lebesgue integrable, but the converse is not necessarily true.

Lemma B.1 *([16, Chapter 9], Proposition 2.16, (c) and (d))*

(i) Let $g \geq 0$ be a monotone non-increasing function; then g is directly Riemann integrable if and only if g is Riemann integrable.

(ii) Let $g \geq 0$ and let ϕ be a distribution function on $\mathbb{R}_+$. If g is directly Riemann integrable, then $\phi * g$ is directly Riemann integrable as well.

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References

[1] Barber, B., and T. Odean (2001): "The Internet and the Investor," *Journal of Economic Perspectives*, 15(1), 41-54.

[2] Barber, B., and T. Odean (2002): "Online Investors: Do the Slow Die First?" *Review of Financial Studies*, 15(2), 455-487.

[3] Bayraktar, E., and H. V. Poor (2004): "Arbitrage in Fractal Modulated Black-Scholes Models When the Volatility is Stochastic," *International Journal of theoretical and Applied Finance*, 8 (3), 1–18.

[4] Bayraktar, E., and H. V. Poor (2004): "Stochastic Differential Games in a Non-Markovian Setting," *SIAM Journal of Control and Optimization*, 43 (5), 1737-1756.

[5] Bayraktar, E., H. V. Poor, and R. Sircar (2004): "Estimating the Fractal Dimension of the S&P 500 Index Using Wavelet Analysis," *International Journal of theoretical and Applied Finance*, 7 (5), 615–643.

[6] Bianchi, S. (2005): "Pathwise Identification of the Memory Function of Multifractional Brownian Motion with Applications to Finance," *International Journal of theoretical and Applied Finance*, 8, 255–281.

[7] Billingsley, P. (1968): *Convergence of Probability Measures*, John Wiley & Sons, New York.

[8] Bingham N. H., C. M. Goldie, and J. L. Teugels (1987): *Regular Variation*, Cambridge University Press, Cambridge.

[9] Brandt, A., P. Franken, and B. Lisek (1990): *Stationary Stochastic Models*, John Wiley & Sons, New York.

[10] Brock, W. A., and C. Hommes (1997): "A Rational Route to Randomness," *Econometrica*, 65 (5), 1059–1095.

[11] Brock, W. A., and C. Hommes (1998): "Heterogeneous Belief and Routes to Chaos in a Simple Asset Pricing Model," *Journal of Economic Dynamics and Control*, 22, 1235–1274.

[12] Cheridito, P. (2001): "Mixed fractional Brownian Motion," *Bernoulli*, 7 (6), 913-934.

[13] Cheridito, P. (2003): "Arbitrage in Fractional Brownian Motion Models," *Finance and Stochastics*, 7(4), 533–553.

[14] Choi, J., D. Laibson, and A. Metrick (2002): "How Does the Internet Affect Trading? Evidence from Investor Behavior in 401(k) Plans," *Journal of Financial Economics*, 64, 397-421.

[15] Çinlar, E. (1975): "Markov Renewal Theory: A Survey," *Management Science*, 21, 727-752.

[16] Çinlar, E. (1975): "*Introduction to Stochastic Processes*," Prentice-Hall, Englewood Cliffs, New Jersey.

[17] Cont, R., and J.-P. Bouchaud (2000): "Herd Behavior and Aggregate Fluctuations in Financial Markets," *Macroeconomic Dynamics*, 4, 170–196.

[18] Cont, R. (2001): Empirical properties of asset returns: stylized facts and statistical issues. *Quantitative Finance*, 1, 223–236.

[19] Cutland, N.J., P.E. Kopp, and W. Willinger (1995): "Stock Price Returns and the Joseph Effect: A Fractional Version of the Black-Scholes Model," *Progress in Probability*, 36, 327-351.

[20] Ding Z., C. W. J. Granger, and R. F. Engle (1993): A long memory property of stock market returns and a new model. *Journal of Empirical Finance*, 1, 83–106.

[21] Dow, J., and S. Werlang (1992): "Uncertainty Aversion, Risk Aversion, and the Optimal Choice of Portfolio," *Econometrica*, 60 (1), 197–204.

[22] Duffie, D., and P. Protter (1992): “From discrete- to continuous Time Finance: Weak Convergence of the Financial Gain Process,” *Mathematical Finance*, 2 (1), 1–15.

[23] Duffield, N.G., and W. Whitt (1998): “A Source Traffic Model and its Transient Analysis for Network Control,” *Stochastic Models* 14, 51–78.

[24] Duffield, N.G., and W. Whitt (1998): “Network Design and Control Using On-Off and Multi-level Source Traffic Models with Heavy Tailed Distributions ,” in *Self-Similar Network Traffic and Performance Evaluation*, K. Park and W. Willinger (eds.), Wiley, Boston, 421–445.

[25] Föllmer, H. (1994): “Stock Price Fluctuations as a Diffusion Model in a Random Environment,” *Phil. Trans. R. Soc. London A*, 374, 471–483.

[26] Föllmer, H., U. Horst, and A. Kirman (2004): “Equilibria in Financial Markets with Heterogeneous Agents: A new Perspective,” *Journal of Mathematical Economics* 41 (1-2), 123-155.

[27] Föllmer, H., and M. Schweizer (1993): “A Microeconomic Approach to Diffusion Models for Stock Prices,” *Mathematical Finance*, 3, 1–23.

[28] Gaigalas, R., and I. Kaj (2003): “Convergence of scaled Renewal Processes and a Packet Arrival Model,” *Bernoulli*, 9, 671–703.

[29] Garman, M. (1976): “Market Microstructure,” *Journal of Financial Economics*, 3, 257–275.

[30] Haliassos, M., and C. C. Bertaut(1995): “Why do so few Hold Stocks,” *The Economic Journal*, 105, 1110-1129.

[31] Heath, D., S. Resnick, and G. Samorodnitsky (1996): “Heavy Tails and Long Range Dependence in On-Off Processes and Associated Fluid Models,” *Mathematics of Operations Research*, 23, 145-165.

[32] Horst, U. (2004): “Financial Price Fluctuations in a Stock Market Model with many Interacting Agents;” *Economic Theory*, 25 (4), 917-932.

[33] Jacod, J., and A. Shiryaev (1987): *Limit Theorems for Stochastic Processes*. Springer-Verlag, Berlin.

[34] Jelenkovic, P. R., and A. A. Lazar (1998): “Subexponential Asymptotics of a Markov-Modulated Random Walk with Queueing Applications,” *Journal of Applied Probability*, 35, 325-347.

[35] Kahneman, D., and A. Tversky (1979): “Prospect Theory: An Analysis of Decision under Risk,” *Econometrica*, 47, 263-291.

[36] Klüppelberg, C., and C. Kühn (2004): “Fractional Brownian Motion as a Weak Limit of Poisson Shot Noise Processes - with Applications to Finance,” *Stochastic Processes and Their Applications*, 113, 333–351.

[37] Kruk, L. (2003): “Functional Limit Theorems for a Simple Auction,” *Mathematics of Operations Research*, 28 (4), 716–751.

[38] Kurtz, T. G., and P. Protter (1991): “Weak Limit Theorems for Stochastic Integrals and Stochastic Differential Equations,” *Annals of Probability*, 19 (3), 1035–1070.

[39] Lin, S. J. (1995): “Stochastic Analysis of Fractional Brownian Motions,” *Stochastics and Stochastics Reports*, 55 , 121–140.

[40] Levy, J.B. and Taqqu, M. S. (2000): “Renewal reward Processes with Heavy-Tailed Inter-Renewal Times and Heay Tailed Rewards,” *Bernoulli*, 6, 23–44.

[41] Lux, T. (1998): “The Socio-Economic Dynamics of Speculative Markets: Interacting Agents, Chaos, and the Fat Tails of Return Distributions,” *Journal of Economic Behavior and Organization*, 33, 143–165.

[42] Lux, T., and M. Marchesi (2000): “Volatility Clustering in Financial Markets: A Microsimulation of Interacting Agents,” *Int. J. Theor. Appl. Finance*, 3 (4), 675–702.

[43] Madrian, B., and D. Shea (2001): “The Power of Suggestion: Inertia in 401(k) Participation and Savings Behavior,” *The Quarterly Journal of Economics*, CXVI(4), 1149-1187.

[44] Mandelbrot, B. B. (1997): *Fractals and Scaling in Finance*, Springer-Verlag, New York.

[45] Mikosch, T., S. Resnick, S. I. Rootzén, and A. Stegeman (2002): “Is Network Traffic Approximated by Stable Levy Motion or Fractional Brownian Motion?,” *Annals of Applied Probability*, 12, 23–68.

[46] New York Stock Exchange (2000): *Shareownership2000*, New York: New York Stock Exchange, *available at http://www.nyse.com/marketinfo/1022949657949.html*

[47] O’Hara, M. (1995): *Market Microstructure Theory*. Blackwell, Massachussetts.

[48] Protter, P. (1990): *Stochastic Integration and Differential Equations*. Springer-Verlag, Berlin.

[49] Rogers, L. C. G. (1997): “Arbitrage with Fractional Brownian Motion,” *Mathematical Finance*, 7 (1), 95–105.

[50] Shefrin, H., and M. Statman (1985): “The Disposition to Sell Winners Too Early and Ride Losers Too Long: Theory and Evidence,” *Journal of Finance*, XL(3), 777-790.

[51] Simonsen, M., and S. Werlang (1991): “Subadditive Probability and Portfilo Inertia,” *Revista de Econometrica*, 11, 1–19.

[52] Sottinen, T. (2001): “Fractional Brownian Motion, Random Walks and Binary Market Models,” *Finance and Stochastics*, 5, 343–355.

[53] Taqqu, M. S., and J. B. Levy (1986): “Using Renewal Processes to Generate Long-Range Dependence and High Variability,” In Dependence in Probability and Statistics, E. Eberlein and M.S. Taqqu, eds., 73–89. Birkhuser; Boston.

[54] Taqqu, M. S., W. Willinger, and R. Sherman (1997): "Proof of a Fundamental Result in Self-Similar Traffic Modeling," *Computer Communications Review*, 27, 5–23.

[55] Whitt, W. (2002): *Stochastic Process Limits: an Introduction to Stochastic-Process Limits and Their Application to Queues*, Springer-Verlag, Berlin.