Interest Rate Modelling and Derivative Pricing

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Part IV

Term Structure Modelling

Outline

What are term structure models compared to Vanilla models?

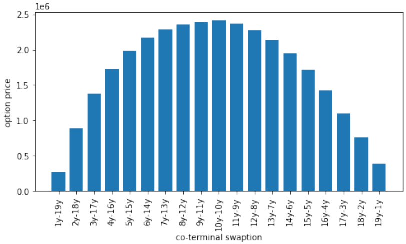
Vanilla models

- Specify dynamics for a single swap rate $S(T)$ with start/end dates T_0/T_n (and details).
- Effectively, only describes terminal distribution of $S(T)$.
- Allows pricing of European swaptions.
- Can be extended to slightly more complex options (with additional assumptions).

Term structure models

- Specify dynamics for evolution of all future zero coupon bonds $P(T, T')$ $(t \leq T \leq T')$.
- Yields (joint) distribution of *all* swap rates $S(T)$.
- Allows pricing of Bermudan swaptions and other complex derivatives.
- Typically, computationally more expensive than Vanilla model pricing.

Why do we need to model the whole term structure of interest rates?



Recall

$$V^{\mathrm{Berm}}(t) = \mathrm{MaxEuropean} + \mathrm{SwitchOption}.$$

- MaxEuropean price is fully determined by Vanilla model.
- Residual SwitchOption price cannot be inferred from Vanilla model.

SwitchOption (i.e. right to postpone future exercise decisions) pricing requires modelling of full interest rate term structure.

Outline

Outline

Heath-Jarrow-Morton specify general dynamics of zero coupon bond prices

Recall our market setting with zero coupon bonds $P(t, T)$ $(t \leq T)$ and bank account $B(t) = \exp\left\{\int_0^t r(s)ds\right\}$.

Discounted bond price is martingale in risk-neutral measure.

Martingale representation theorem yields

$$d\left(\frac{P(t,T)}{B(t)}\right) = -\frac{P(t,T)}{B(t)} \cdot \sigma_P(t,T)^\top \cdot dW(t)$$

where $\sigma_P(t,T) = \sigma_P(t,T,\omega)$ is a d-dimensional process adapted to $\mathcal{F}_t$. We also impose $\sigma_P(T,T) = 0$ (pull-to-par for bond prices with $P(T,T) = 1$).

- What are dynamics of (un-discounted) zero bonds $P(t, T)$?
- What are dynamics of forward rates $f(t, T)$?
- How to specify bond price volatility?

What are dynamics of zero bonds $P(t, T)$?

Lemma (Bond price dynamics)

Under the risk-neutral measure zero bond prices evolve according to

$$\frac{dP(t,T)}{P(t,T)} = r(t) \cdot dt - \sigma_P(t,T)^\top \cdot dW(t).$$

Proof.

Apply Ito's lemma to $d\left(P(t,T)/B(t)\right)$ and compare with dynamics of discounted bond prices. □

- Zero bond drift equals short rate $r(t)$.
- Zero bond volatility $\sigma_P(t,T)$ remains unchanged.
- How do we get $r(t)$?

What are dynamics of forward rates $f(t, T)$?

Theorem (Forward rate dynamics)

Consider a d-dimensional forward rate volatility process $\sigma_f(t, T) = \sigma_f(t, T, \omega)$ adapted to $\mathcal{F}_t$. Under the risk-neutral measure the dynamics of forward rates $f(t, T)$ are given by

$$df(t, T) = \sigma_f(t, T)^\top \cdot \left[\int_t^T \sigma_f(t, u) du \right] \cdot dt + \sigma_f(t, T)^\top \cdot dW(t)$$

and $f(0, T) = f^M(0, T)$. Moreover

$$\sigma_P(t, T) = \int_t^T \sigma_f(t, u) du.$$

- Once volatility $\sigma_f(t, T)$ is specified no-arbitrage pricing yields the drift.
- Model is auto-calibrated to initial yield curve via $f(0, T) = f^M(0, T)$.

We prove the forward rate dynamics (1/2)

Recall

$$f(t,T) = -\frac{\partial}{\partial T}\ln\left(P(t,T)\right).$$

Exchanging order of differentiation yields

$$df(t,T) = d\left[-\frac{\partial}{\partial T}\ln\left(P(t,T)\right)\right] = -\frac{\partial}{\partial T} d\ln\left(P(t,T)\right).$$

Applying Ito's lemma (to $d\ln\left(P(t,T)\right)$) with bond price dynamics yields

$$\begin{aligned} d\ln\left(P(t,T)\right) &= \frac{d(P(t,T))}{P(t,T)} - \frac{\sigma_P(t,T)^\top \sigma_P(t,T)}{2}\cdot dt \\ &= \left[r(t) - \frac{\sigma_P(t,T)^\top \sigma_P(t,T)}{2}\right]\cdot dt - \sigma_P(t,T)^\top \cdot dW(t). \end{aligned}$$

Differentiating $d\ln\left(P(t,T)\right)$ w.r.t. T gives

$$df(t,T) = \left[\frac{\partial}{\partial T}\sigma_P(t,T)\right]^\top \sigma_P(t,T)\cdot dt + \left[\frac{\partial}{\partial T}\sigma_P(t,T)\right]^\top \cdot dW(t).$$

We prove the forward rate dynamics (2/2)

$$df(t,T) = \left[\frac{\partial}{\partial T}\sigma_P(t,T)\right]^\top \sigma_P(t,T)\cdot dt + \left[\frac{\partial}{\partial T}\sigma_P(t,T)\right]^\top \cdot dW(t).$$

Denote

$$\sigma_f(t,T) = \frac{\partial}{\partial T}\sigma_P(t,T).$$

With terminal condition $\sigma_P(T,T) = 0$ follows integral representation

$$\sigma_P(t,T) = \int_t^T \sigma_f(t,u)du.$$

Substituting back gives the result

$$df(t,T) = \sigma_f(t,T)^\top \cdot \left[\int_t^T \sigma_f(t,u)du\right]\cdot dt + \sigma_f(t,T)^\top \cdot dW(t).$$

It will be useful to have the dynamics under the forward measure as well

Lemma (Brownian motion in T-forward measure)

Consider our HJM framework with Brownian motion $W(t)$ under the risk-neutral measure and

$$\frac{dP(t,T)}{P(t,T)} = r(t) \cdot dt - \sigma_P(t,T)^\top \cdot dW(t).$$

Under the T-forward measure the bond price dynamics are

$$\frac{dP(t,T)}{P(t,T)} = \left[r(t) + \sigma_P(t,T)^\top \sigma_P(t,T) \right] \cdot dt - \sigma_P(t,T)^\top \cdot dW^T(t)$$

with $W^T(t)$ a Brownian motion (under the T-forward measure). Moreover,

$$dW^T(t) = \sigma_P(t,T) \cdot dt + dW(t).$$

T-forward measure dynamics can be shown by Ito's lemma (1/2)

Abbrev. deflated bond prices $Y(t) = \frac{P(t,T)}{B(t)}$, then $\frac{dY(t)}{Y(t)} = -\sigma_P(t,T)^\top dW(t)$.

Now consider $1/Y(t)$ and apply Ito's lemma

$$d\left(\frac{1}{Y(t)}\right) = -\frac{dY(t)}{Y(t)^2} + \frac{1}{2}\frac{2}{Y(t)^3}\left[dY(t)\right]^2 = \frac{1}{Y(t)}\left[\left(\frac{dY(t)}{Y(t)}\right)^2 - \frac{dY(t)}{Y(t)}\right]$$
$$= \frac{1}{Y(t)}\left[\sigma_P(t,T)^\top \sigma_P(t,T)dt + \sigma_P(t,T)^\top dW(t)\right]$$
$$= \frac{\sigma_P(t,T)^\top}{Y(t)}\left[\sigma_P(t,T)dt + dW(t)\right].$$

T-forward measure dynamics can be shown by Ito's lemma (2/2)

However, $1/Y(t) = B(t)/P(t, T)$ is a martingale in T-forward measure and $d\left(\frac{1}{Y(t)}\right)$must be drift-less in T-forward measure.

Define $W^T(t)$ with

$$dW^T(t) = \sigma_P(t, T)dt + dW(t).$$

Then $W^T(t)$ must be a Brownian motion in the T-forward measure.

Substituting $dW(t)$ in the risk-neutral bond price dynamics finally gives the dynamics under T-forward measure.

Outline

Short rate can be derived from forward rate dynamics

Corollary (Short rate specification)

In our HJM framework the short rate becomes

$$
\begin{aligned}
r(t) &= f(t,t) \\
&= f(0,t) + \\
&\quad \int_0^t \sigma_f(u,t)^\top \cdot \left[\int_u^t \sigma_f(u,s) ds \right] du + \int_0^t \sigma_f(u,t)^\top \cdot dW(u).
\end{aligned}
$$

Proof.

Follows directly from forward rate dynamics and integration from 0 to t. □

- Note that integrand in diffusion term $D(t) = \int_0^t \sigma_f(u,t)^\top \cdot dW(u)$ depends on t.
- In general, $D(t)$ is *not* a martingale.
- In general, $r(t)$ is *not* Markovian unless volatility $\sigma_f(t,T)$ is suitably restricted.

We analyse diffusion term in detail

$$D(t) = \int_0^t \sigma_f(u,t)^\top \cdot dW(u).$$

It follows

$$\begin{aligned}
D(T) &= \int_0^t \sigma_f(u,T)^\top \cdot dW(u) + \int_t^T \sigma_f(u,T)^\top \cdot dW(u) \\
&= D(t) + \int_t^T \sigma_f(u,T)^\top \cdot dW(u) \\
&\qquad + \int_0^t \sigma_f(u,T)^\top \cdot dW(u) - \int_0^t \sigma_f(u,t)^\top \cdot dW(u) \\
&= D(t) + \int_t^T \sigma_f(u,T)^\top \cdot dW(u) + \int_0^t \left[\sigma_f(u,T) - \sigma_f(u,t)\right]^\top \cdot dW(u).
\end{aligned}$$

- How is $\mathbb{E}^{\mathbb{Q}}\left[D(T) \,|\, D(t)\right]$ (knowing only last state) related to $\mathbb{E}^{\mathbb{Q}}\left[D(T) \,|\, \mathcal{F}_t\right]$ (knowing full history)?
- If D is Markovian then $\mathbb{E}^{\mathbb{Q}}\left[D(T) \,|\, D(t)\right] = \mathbb{E}^{\mathbb{Q}}\left[D(T) \,|\, \mathcal{F}_t\right]$ (neccessary condition).

Compare $\mathbb{E}^{\mathbb{Q}}\left[D(T) \,|\, D(t)\right]$ and $\mathbb{E}^{\mathbb{Q}}\left[D(T) \,|\, \mathcal{F}_t\right]$ (1/2)

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}\left[D(T) \,|\, \mathcal{F}_t\right] &= \mathbb{E}^{\mathbb{Q}}\left[D(t) + \int_t^T \sigma_f(u,T)^\top dW(u) \,\Big|\, \mathcal{F}_t\right] \\
&\quad + \mathbb{E}^{\mathbb{Q}}\left[\int_0^t \left[\sigma_f(u,T) - \sigma_f(u,t)\right]^\top dW(u) \,\Big|\, \mathcal{F}_t\right] \\
&= D(t) + 0 + \underbrace{\int_0^t \left[\sigma_f(u,T) - \sigma_f(u,t)\right]^\top dW(u)}_{I(t,T)}.
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}\left[D(T) \,|\, D(t)\right] &= \mathbb{E}^{\mathbb{Q}}\left[D(t) + \int_t^T \sigma_f(u,T)^\top dW(u) \,\Big|\, D(t)\right] \\
&\quad + \mathbb{E}^{\mathbb{Q}}\left[\int_0^t \left[\sigma_f(u,T) - \sigma_f(u,t)\right]^\top dW(u) \,\Big|\, D(t)\right] \\
&= D(t) + 0 + \mathbb{E}^{\mathbb{Q}}\left[\int_0^t \left[\sigma_f(u,T) - \sigma_f(u,t)\right]^\top dW(u) \,\Big|\, D(t)\right].
\end{aligned}$$

Compare $\mathbb{E}^{\mathbb{Q}}[D(T) \mid D(t)]$ and $\mathbb{E}^{\mathbb{Q}}[D(T) \mid \mathcal{F}_t]$ (2/2)

$$\mathbb{E}^{\mathbb{Q}}[D(T) \mid \mathcal{F}_t] = D(t) + \underbrace{\int_0^t [\sigma_f(u,T) - \sigma_f(u,t)]^\top \, dW(u)}_{I(t,T)}.$$

$$\mathbb{E}^{\mathbb{Q}}[D(T) \mid D(t)] = D(t) + \mathbb{E}^{\mathbb{Q}}\left[\int_0^t [\sigma_f(u,T) - \sigma_f(u,t)]^\top \, dW(u) \mid D(t)\right].$$

- $\mathbb{E}^{\mathbb{Q}}[D(T) \mid D(t)] = \mathbb{E}^{\mathbb{Q}}[D(T) \mid \mathcal{F}_t]$ only if $I(t,T)$ is non-random or deterministic function of $D(t)$.

An important separability condition makes $D(t)$ Markovian

Assume

$$\sigma_f(t, T) = g(t) \cdot h(T)$$

with $g(t)$ (scalar) process adapted to $\mathcal{F}_t$ and $h(T)$ (scalar) deterministic and differentiable function.
Then

$$\begin{aligned} D(T) &= \int_0^t g(u) \cdot h(T) \cdot dW(u) + \int_t^T g(u) \cdot h(T) \cdot dW(u) \\ &= \frac{h(T)}{h(t)} \cdot D(t) + h(T) \cdot \int_t^T g(u) \cdot dW(u). \end{aligned}$$

Thus

$$\mathbb{E}^{\mathbb{Q}}\left[D(T) \,|\, D(t)\right] = \mathbb{E}^{\mathbb{Q}}\left[D(T) \,|\, \mathcal{F}_t\right] = \frac{h(T)}{h(t)} \cdot D(t).$$

Moreover

$$d\left(D(t)\right) = \frac{h'(t)}{h(t)} \cdot D(t) \cdot dt + g(t) \cdot h(t) \cdot dW(t).$$

Outline

We describe a very general but still tractable class of models

- We give a general description of a class of term structure models.
- Typically, these models are called Cheyette-type or quasi-Gaussian models; also associated with work by Ritchken and Sankarasubramanian (1995).
- Particular parameter choices will specialise general model to classical model (e.g. Hull-White model).
- More complex parameter choices yield powerful model instances for complex interest rate derivatives.

Quasi-Gaussian models are important models in the industry.

Separable forward rate volatility

Definition (Separable forward rate volatility)

The forward rate volatility $\sigma_f(t, T)$ of an HJM model is considered of separable form if

$$\sigma_f(t, T) = g(t)h(T)$$

for a matrix-valued process $g(t) = g(t, \omega) \in \mathbb{R}^{d \times d}$ adapted to $\mathcal{F}_t$ and a vector-valued deterministic function $h(T) \in \mathbb{R}^d$.

Corollary

For a separable forward rate volatility $\sigma_f(t, T) = g(t)h(T)$ the bond price volatility $\sigma_P(t, T)$ becomes

$$\sigma_P(t, T) = g(t) \int_t^T h(u)du.$$

Forward rate representation follows directly

Lemma

For a separable forward rate volatility $\sigma_f(t,T) = g(t)h(T)$ the forward rate becomes

$$f(t,T) = f(0,T) + h(T)^\top \int_0^t g(s)^\top g(s) \left(\int_s^T h(u)du \right) ds + h(T)^\top \int_0^t g(s)^\top dW(s)$$

and

$$r(t) = f(0,t) + h(t)^\top \left[\int_0^t g(s)^\top g(s) \left(\int_s^t h(u)du \right) ds + \int_0^t g(s)^\top dW(s) \right].$$

Proof.

Follows directly from definition. $\square$

We need to introduce new state variables to derive Markovian representation of short rate

Re-write $h(t)^\top = \mathbf{1}^\top H(t)$ and

$$r(t) = f(0,t) + \mathbf{1}^\top H(t) \left[\int_0^t g(s)^\top g(s) \left(\int_s^t h(u) du \right) ds + \int_0^t g(s)^\top dW(s) \right]$$

with

$$\mathbf{1} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \text{ and } H(t) = diag\left(h(t)\right) = \begin{pmatrix} h_1(t) & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & h_d(t) \end{pmatrix}.$$

Introduce vector of state variables $x(t)$ with

$$x(t) = H(t) \left[\int_0^t g(s)^\top g(s) \left(\int_s^t h(u) du \right) ds + \int_0^t g(s)^\top dW(s) \right].$$

We derive the dynamics of the short rate

Theorem (Separable HJM short rate dynamics)

In an HJM model with separable volatility the short rate is given by $r(t) = f(0,t) + \mathbf{1}^\top x(t)$. *The vector of state variables* $x(t)$ *evolves according to* $x(0) = 0$ *and*

$$dx(t) = [y(t)\mathbf{1} - \chi(t)x(t)]\,dt + H(t)g(t)^\top dW(t)$$

with symmetric matrix of auxilliary variables $y(t)$ *as*

$$y(t) = H(t)\left(\int_0^t g(s)^\top g(s)ds\right)H(t)$$

and diagonal matrix of mean reversion parameters $\chi(t)$ *as*

$$\chi(t) = -\frac{dH(t)}{dt}H(t)^{-1}.$$

Proof follows straight forward via differentiation (1/3)

We have

$$x(t) = H(t) \underbrace{\left[\int_0^t g(s)^\top g(s) \left(\int_s^t h(u) du \right) ds + \int_0^t g(s)^\top dW(s) \right]}_{G(t)}.$$

$$\begin{aligned} dx(t) &= H'(t) \cdot G(t) \cdot dt + H(t) \cdot dG(t) \\ &= H'(t) \cdot H(t)^{-1} \cdot H(t) \cdot G(t) \cdot dt + H(t) \cdot dG(t) \\ &= -\chi(t) \cdot x(t) \cdot dt + H(t) \cdot dG(t). \end{aligned}$$

Proof follows straight forward via differentiation (2/3)

$$
\begin{aligned}
dx(t) &= -\chi(t) \cdot x(t) \cdot dt + H(t) \cdot dG(t), \\
G(t) &= \int_0^t g(s)^\top g(s) \left(\int_s^t h(u) du \right) ds + \int_0^t g(s)^\top dW(s).
\end{aligned}
$$

Leibnitz rule yields

$$
\begin{aligned}
dG(t) &= \left[g(t)^\top g(t) \left(\int_t^t h(u) du \right) + \int_0^t g(s)^\top g(s) \frac{d}{dt} \left(\int_s^t h(u) du \right) ds \right] dt \\
&\quad + g(t)^\top dW(t) \\
&= \left[0 + \int_0^t g(s)^\top g(s) \cdot H(t)\mathbf{1} \cdot ds \right] dt + g(t)^\top dW(t) \\
&= \left[\left(\int_0^t g(s)^\top g(s) ds \right) H(t)\mathbf{1} \right] dt + g(t)^\top dW(t).
\end{aligned}
$$

Proof follows straight forward via differentiation (3/3)

Combining results gives

$$
\begin{aligned}
dx(t) &= -\chi(t)\cdot x(t)\cdot dt + H(t)\cdot dG(t) \\
&= \left[H(t)\left(\int_0^t g(s)^\top g(s)ds\right)H(t)\mathbf{1} - \chi(t)\cdot x(t)\right]dt \\
&\quad + H(t)\cdot g(t)^\top dW(t) \\
&= \left[y(t)\cdot\mathbf{1} - \chi(t)\cdot x(t)\right]dt + H(t)\cdot g(t)^\top dW(t).
\end{aligned}
$$

- Note that $dx(t)$ depends on accumulated previous volatility via $\int_0^t g(s)^\top g(s)ds$.
- $x(t)$ is Markovian only if volatility function $g(t)$ is deterministic.
- In general, short rate dynamics can be ammended by dynamics of $y(t)$.

Short rate dynamics can be written in terms of state and auxilliary variables (1/2)

Corollary (Augmented short rate dynamics)

In an HJM model with separable volatility the short rate is given via $r(t) = f(0,t) + \mathbf{1}^\top x(t)$ *with*

$$
\begin{aligned}
dx(t) &= \left[y(t)\cdot\mathbf{1} - \chi(t)\cdot x(t)\right]dt + \sigma_r(t)^\top dW(t),\\
dy(t) &= \left[\sigma_r(t)^\top\sigma_r(t) - \chi(t)y(t) - y(t)\chi(t)\right]dt,
\end{aligned}
$$

and $x(0) = 0$, $y(0) = 0$.

Short rate dynamics can be written in terms of state and auxilliary variables (2/2)

Proof.
Set $\sigma_r(t) = g(t)H(t)$ and differentiate
$y(t) = H(t) \left(\int_0^t g(s)^\top g(s) ds \right) H(t)$. $\square$

- Model class also called Cheyette or quasi-Gaussian models.
- Typically $\sigma_r(t)$ and $\chi(t)$ are specified and $\sigma_f(t, T)$ is reconstructed via

$$
\begin{aligned}
H'(t) =& -\chi(t)H(t),\ H(0) = 1 \quad \text{and} \\
g(t) =& \sigma_r(t)H(t)^{-1}.
\end{aligned}
$$

Forward rates and zero bonds can be written in terms of state/auxilliary variables

Theorem (Forward rate and zero bond reconstruction)

In our HJM model setting we get

$$f(t,T) = f(0,T) + \mathbf{1}^\top H(T)H(t)^{-1}\left[x(t) + y(t)G(t,T)\right]$$

and

$$P(t,T) = \frac{P(0,T)}{P(0,t)} \exp\left\{-G(t,T)^\top x(t) - \frac{1}{2}G(t,T)^\top y(t) G(t,T)\right\}$$

with

$$G(t,T) = \int_t^T H(u)H(t)^{-1}\mathbf{1}du.$$

- We prove the first part for $f(t,T)$.
- And we sketch the proof for the second part for $P(t,T)$.

We prove the first part for $f(t, T)$ (1/2)...

$$\underbrace{\mathbf{1}^\top H(T)H(t)^{-1}x(t)}_{I_1}$$

$$= h(T)^T \left[\int_0^t g(s)^\top g(s) \left(\int_s^{t} h(u)du\right) ds + \int_0^t g(s)^\top dW(s)\right].$$

$$\underbrace{\mathbf{1}^\top H(T)H(t)^{-1}y(t)G(t, T)}_{I_2}$$

$$= h(T)^\top \left(\int_0^t g(s)^\top g(s)ds\right) \int_{t}^T h(u)du.$$

We prove the first part for $f(t, T)$ (2/2)...

$$
\begin{aligned}
& I_1 + I_2 \\
& = h(T)^\top \times \\
& \quad \left[\int_0^t g(s)^\top g(s) \left(\int_s^t h(u) du \right) ds + \left(\int_0^t g(s)^\top g(s) ds \right) \int_t^T h(u) du \right] \\
& \quad + h(T)^T \int_0^t g(s)^\top dW(s) \\
& = h(T)^T \times \\
& \quad \left[\int_0^t g(s)^\top g(s) \left(\int_s^t h(u) du + \int_t^T h(u) du \right) ds + \int_0^t g(s)^\top dW(s) \right] \\
& = h(T)^T \left[\int_0^t g(s)^\top g(s) \left(\int_s^T h(u) du \right) ds + \int_0^t g(s)^\top dW(s) \right] \\
& = f(t, T) - f(0, T)
\end{aligned}
$$

... and sketch the proof for the second part for $P(t, T)$ (1/2)

$$
\begin{aligned}
P(t,T) &= \exp\left\{-\int_t^T f(t,s)ds\right\} \\
&= \exp\left\{-\int_t^T \left(f(0,s) + \mathbf{1}^\top H(s)H(t)^{-1}\left[x(t) + y(t)G(t,s)\right]\right) ds\right\} \\
&= \frac{P(0,T)}{P(0,t)} \cdot \exp\left\{-\underbrace{\left(\int_t^T \mathbf{1}^\top H(s)H(t)^{-1} ds\right)}_{G(t,T)^\top} x(t)\right\} \cdot \\
&\qquad \exp\left\{-\int_t^T \mathbf{1}^\top H(s)H(t)^{-1} y(t)G(t,s)ds\right\}
\end{aligned}
$$

... and sketch the proof for the second part for $P(t, T)$ (2/2)

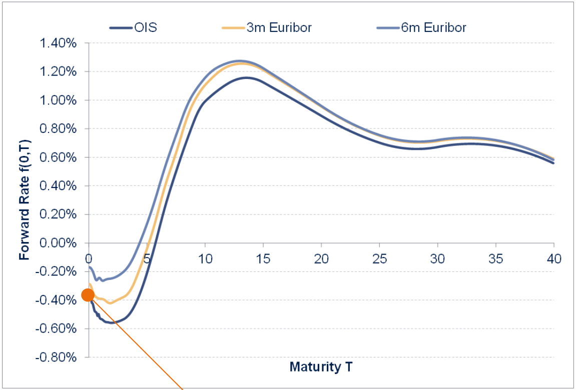
It remains to show that

$$\int_t^T \mathbf{1}^\top H(s) H(t)^{-1} y(t) G(t,s) ds = \frac{1}{2} G(t,T)^\top y(t) G(t,T).$$

We note that both sides of above equation are zero for $T = t$.
The equality for $T > t$ follows then by differentiating both sides w.r.t. T and comparing terms.

Outline

We take a complementary view to HJM framework and consider direct modelling of the short rate $r(t)$



short rate $r(t) = f(t, t)$

We model short rate of the discount curve as offset point for future rates.

Short rate suffices to specify evolution of the full yield curve

Recall zero bond formula

$$P(t, T) = \mathbb{E}^{\mathbb{Q}}\left[\exp\left\{-\int_t^T r(s)ds\right\} \mid \mathcal{F}_t\right].$$

- Once dynamics of $r(t)$ are specified all zero bonds can be derived.

Libor rates (in multi-curve setting) are

$$L(t; T_0, T_1) = \mathbb{E}^{T_1}\left[L(T; T_0, T_1) \mid \mathcal{F}_t\right] = \left[\frac{P(t, T_0)}{P(t, T_1)} \cdot D(T_0, T_1) - 1\right]\frac{1}{\tau}.$$

- With zero bonds $P(t, T)$ (and tenor basis factors $D(T_0, T_1)$) we can also derive future Libor rates.

Short rate is a natural choice of state variable for modelling evolution of interest rates.

Outline

Vasicek model and Ho-Lee model were the first models for the short rate

Vasicek (1977) assumed Ornstein-Uhlenbeck process

$$dr(t) = \kappa\,(\theta - r(t))\,dt + \sigma dW(t), \quad r(0) = r_0$$

for positive constants r_0, κ, θ, and σ.

- Model is not too different from HJM model representation.
- Constant parameters (in particular θ) limit ability to reproduce/calibrate yield curve observed today.

Ho and Lee (1986) introduce exogenous time-dependent drift parameter,

$$dr(t) = \theta(t)dt + \sigma dW(t).$$

- Drift parameter $\theta(t)$ is used to match today's zero bonds $P(0, T)$.
- Lack of mean reversion is considered main disadvantage.
- Model was historically used with binomial tree implementation.

Hull and White (1990) extended Vasicek model by $\theta(t)$

Definition (Hull-White model)

In the Hull-White model the short rate evolves according to

$$dr(t) = [\theta(t) - a(t)r(t)]\,dt + \sigma(t)dW(t)$$

with deterministic scalar functions $\theta(t)$, $a(t)$, and $\sigma(t) > 0$.

- $\theta(t)$ is mean reversion level,
- $a(t)$ is mean reversion speed, and
- $\sigma(t)$ is short rate volatility.

- Original reference is J. Hull and A. White. Pricing interest-rate-derivative securities. *The Review of Financial Studies*, 3:573–592, 1990
- To simplify analytical tractability we will assume
  - constant mean reversion speed $a(t) = a > 0$, and
  - piece-wise constant short rate volatility function on a siutable time grid $\{t_0, \ldots, t_k\}$,

$$\sigma(t) = \sum_{i=1}^{k} \mathbb{1}_{\{t_{i-1} \le t < t_i\}} \cdot \sigma_i.$$

How do we calibrate the drift $\theta(t)$?

Lemma (Hull-White drift calibration)

In the risk-neutral specification of the Hull-White model the drift term $\theta(t)$ is given by

$$\theta(t) = \frac{\partial}{\partial T} f(0,t) + a \cdot f(0,t) + \int_0^t \left[e^{-a(t-u)} \sigma(u) \right]^2 du.$$

Here $f(0,t) = f^M(0,t)$ is exogenously specified and assumed continuously differentiable w.r.t. the maturity T.

Proof follows along the following steps

- Calculate $r(s)$ via integration.
- Integrate $I(t,T) = \int_t^T r(s) ds$ and calculate distribution of $I(t,T)$.[5]
- Derive $\theta(t)$ such that $\mathbb{E}^{\mathbb{Q}}\left[e^{-I(0,t)} \right] = P(0,T)$.

[5] We will re-use distribution of integrated short rate $I(t,T)$ later for options on compounded rates.

Proof (1/4) - calculate $r(s)$

We show that for $s \geq t$

$$r(s) = e^{-a(s-t)} \left[r(t) + \int_t^s e^{a(u-t)} \left[\theta(u)du + \sigma(u)dW(u) \right] \right].$$

$$\begin{aligned} dr(s) &= -ar(s)ds + e^{-a(s-t)} \left[e^{a(s-t)} \left[\theta(s)ds + \sigma(s)dW(s) \right] \right] \\ &= \left[\theta(s) - ar(s) \right] ds + \sigma(s)dW(s). \end{aligned}$$

Use notation $[\cdot]'(t,T) = \frac{\partial}{\partial T}[\cdot]$. Set $I(t,T) = \int_t^T r(s)ds$, then $I'(t,T) = \frac{\partial I(t,T)}{\partial T} = r(T)$. We show

$$I(t,T) = G(t,T)r(t) + \int_t^T G(u,T) \left[\theta(u)du + \sigma(u)dW(u) \right]$$

with

$$G(t,T) = \int_t^T e^{-a(u-t)} du = \left[\frac{1 - e^{-a(T-t)}}{a} \right].$$

Proof (2/4) - calculate distribution $I(t, T)$

$$
\begin{aligned}
I(t, T) &= G(t, T)r(t) + \int_t^T G(u, T)\left[\theta(u)du + \sigma(u)dW(u)\right], \\
I'(t, T) &= G'(t, T)r(t) + 0 + \int_t^T G'(u, T)\left[\theta(u)du + \sigma(u)dW(u)\right] \\
&= e^{-a(T-t)}r(t) + \int_t^T e^{-a(T-u)}\left[\theta(u)du + \sigma(u)dW(u)\right] \\
&= e^{-a(T-t)}\left[r(t) + \int_t^T e^{a(u-t)}\left[\theta(u)du + \sigma(u)dW(u)\right]\right] \\
&= r(T).
\end{aligned}
$$

Conditional on $\mathcal{F}_t$, integral is normally distributed, $I(t, T)|_{\mathcal{F}_t} \sim N(\mu, \sigma^2)$ with

$$
\begin{aligned}
\mu(t, T) &= G(t, T)r(t) + \int_t^T G(u, T)\theta(u)du, \\
\sigma(t, T)^2 &= \int_t^T \left[G(u, T)\sigma(u)\right]^2 du.
\end{aligned}
$$

Proof (3/4) - calculate forward rate

$I(t,T)|_{\mathcal{F}_t} \sim N(\mu, \sigma^2)$ with

$$\mu(t,T) = G(t,T)r(t) + \int_t^T G(u,T)\theta(u)du,$$

$$\sigma^2(t,T) = \int_t^T \left[G(u,T)\sigma(u)\right]^2 du.$$

$$P(t,T) = \mathbb{E}^{\mathbb{Q}}\left[e^{-I(t,T)} \,|\, \mathcal{F}_t\right] = e^{-\mu(t,T)+\frac{1}{2}\sigma^2(t,T)}.$$

$$\begin{aligned} f(t,T) &= -\frac{\partial}{\partial T}\ln\left[P(t,T)\right] = \frac{d}{dT}\left[\mu(t,T) - \frac{1}{2}\sigma^2(t,T)\right] \\ &= G'(t,T)r(t) + 0 + \int_t^T G'(u,T)\theta(u)du \\ &\quad - \frac{1}{2}\left[0 + \int_t^T 2G(u,T)G'(u,T)\sigma(u)^2 du\right] \\ &= G'(t,T)r(t) + \int_t^T G'(u,T)\theta(u)du - \int_t^T G'(u,T)G(u,T)\sigma(u)^2 du. \end{aligned}$$

Proof (4/4) - derive drift $\theta(t)$

$$f(t,T) = G'(t,T)r(t) + \int_t^T G'(u,T)\theta(u)du - \int_t^T G'(u,T)G(u,T)\sigma(u)^2 du.$$

Use $G'(t,T) = e^{-a(T-t)}$ and $G''(t,T) = -aG'(t,T)$, then

$$\begin{aligned} f'(t,T) &= G''(t,T)r(t) + \theta(T) + \int_t^T G'(u,T)\theta(u)du - 0 \\ &\quad - \int_t^T \left[G''(u,T)G(u,T) + G'(u,T)^2\right]\sigma(u)^2 du \\ &= \theta(T) - af(t,T) - \int_t^T \left[G'(u,T)\sigma(u)\right]^2 du. \end{aligned}$$

This finally gives the result (with $t=0$)

$$\begin{aligned} \theta(T) &= f'(t,T) + af(t,T) + \int_t^T \left[G'(u,T)\sigma(u)\right]^2 du \\ &= f'(0,T) + af(0,T) + \int_0^T \left[e^{-a(T-u)}\sigma(u)\right]^2 du. \end{aligned}$$

Do we really need the drift $\theta(t)$?

- Risk-neutral drift representation

$$\theta(t) = \frac{\partial}{\partial T} f(0,t) + a \cdot f(0,t) + \int_0^t \left[e^{-a(t-u)} \sigma(u) \right]^2 du$$

  poses some obstacles.

- Derivative $\frac{\partial}{\partial T} f(0,t)$ may cause numerical difficulties.
- In some market situations you want to have jumps in $f(0,t)$.
- This is relevant in particular for the short end of OIS curve.

- Fortunately, for most applications we don't need drift term.

- HJM representation allows avoiding it alltogether.

Now we can also derive future zero bond prices I

Theorem (Zero bonds in Hull-White model)

In the Hull-White model future zero bond prices are given by

$$P(t,T) = \frac{P(0,T)}{P(0,t)} \cdot \exp\left\{ -G(t,T)\left[r(t) - f(0,t)\right] - \frac{G(t,T)^2}{2} \int_0^t \left[e^{-a(t-u)} \sigma(u) \right]^2 du \right\}$$

with

$$G(t,T) = \int_t^T e^{-a(u-t)} du = \left[\frac{1 - e^{-a(T-t)}}{a} \right].$$

- The proof is a bit technical.
- We already derived the zero bond representation

$$P(t,T) = \mathbb{E}^{\mathbb{Q}}\left[e^{-I(t,T)} \mid \mathcal{F}_t \right] = e^{-\mu(t,T) + \frac{1}{2}\sigma^2(t,T)}.$$

Now we can also derive future zero bond prices II

We have from the proof of risk-neutral drift that

$$f(t,T) = G'(t,T)r(t) + \int_t^T G'(u,T)\theta(u)du - \int_t^T G'(u,T)G(u,T)\sigma^2(u)du$$

and

$$P(t,T) = e^{-G(t,T)r(t) - \int_t^T G(u,T)\theta(u)du + \frac{1}{2}\int_t^T G(u,T)^2\sigma^2(u)du}.$$

We aim at calculating the term

$$I(t,T) = -\int_t^T G(u,T)\theta(u)du + \frac{1}{2}\int_t^T G(u,T)^2\sigma^2(u)du.$$

Now we can also derive future zero bond prices III

Consider

$$
\begin{aligned}
&\log\left(\frac{P(0,t)}{P(0,T)}\right)\\
&= \left[G(0,T) - G(0,t)\right] r(0)\\
&\quad + \int_0^T G(u,T)\theta(u)du - \int_0^t G(u,t)\theta(u)du\\
&\quad - \frac{1}{2}\left[\int_0^T G(u,T)^2\sigma^2(u)du - \int_0^t G(u,t)^2\sigma^2(u)du\right]\\
&= \left[G(0,T) - G(0,t)\right] r(0)\\
&\quad + \int_t^T G(u,T)\theta(u)du + \int_0^t \left[G(u,T) - G(u,t)\right]\theta(u)du\\
&\quad - \frac{1}{2}\left[\int_t^T G(u,T)^2\sigma^2(u)du + \int_0^t \left[G(u,T)^2 - G(u,t)^2\right]\sigma^2(u)du\right].
\end{aligned}
$$

Now we can also derive future zero bond prices IV

We use $G(u,T) - G(u,t) = G(t,T)G'(u,t)$ and re-arrange terms. Then

$$\begin{aligned} I(t,T) = \log\left(\frac{P(0,T)}{P(0,t)}\right) &+ G(t,T)G'(0,t)r(0) \\ &+ G(t,T)\int_0^t G'(u,t)\theta(u)du \\ &- \frac{1}{2}\int_0^t \underbrace{\left[G(u,T)+G(u,t)\right]\left[G(u,T)-G(u,t)\right]}_{[G(u,T)-G(u,t)+2G(u,t)]G(t,T)G'(u,t)}\sigma^2(u)du. \end{aligned}$$

We use representation for forward rate $f(t,T)$ and get

$$\begin{aligned} I(t,T) &= \log\left(\frac{P(0,T)}{P(0,t)}\right) + G(t,T)f(0,t) \\ &\quad - \frac{1}{2}\int_0^t \left[G(u,T)-G(u,t)\right]G(t,T)G'(u,t)\sigma^2(u)du \\ &= \log\left(\frac{P(0,T)}{P(0,t)}\right) + G(t,T)f(0,t) - \frac{G(t,T)^2}{2}\int_0^t G'(u,t)^2\sigma^2(u)du. \end{aligned}$$

Now we can also derive future zero bond prices V

Finally, we get the result

$$
\begin{aligned}
P(t,T) &= e^{-G(t,T)r(t)+I(t,T)} \\
&= \frac{P(0,T)}{P(0,t)} e^{-G(t,T)[r(t)-f(0,t)]-\frac{G(t,T)^2}{2}\int_0^t \left[e^{-a(t-u)}\sigma(u)\right]^2 du}.
\end{aligned}
$$

- Future zero coupon bonds depend on:
  - today's yield curve $f(0,t)$,
  - mean reversion parameter a via $G(t,T)$, and
  - short rate volatility $\sigma(t)$.
- We see that drift $\theta(t)$ is not required for future zero coupon bonds.

Outline

Recall short rate dynamics in separable HJM model

We consider a one-factor model ($d = 1$)

$$\begin{aligned} r(t) &= f(0,t) + x(t) \\ dx(t) &= [y(t) - \chi(t) \cdot x(t)]\, dt + \sigma_r(t) \cdot dW(t) \\ dy(t) &= \left[\sigma_r(t)^2 - 2 \cdot \chi(t) \cdot y(t)\right] \cdot dt \end{aligned}$$

with

$$H'(t) = -\chi(t) H(t),\; H(0) = 1 \text{ and } g(t) = H(t)^{-1} \sigma_r(t).$$

- How does this relate to Hull-White model with

$$dr(t) = [\theta(t) - a \cdot r(t)] \cdot dt + \sigma(t) \cdot dW(t)?$$

Differentiate short rate in HJM model

$$
\begin{aligned}
dr(t) &= f'(0,t)dt + dx(t) \\
&= f'(0,t)dt + \left[y(t) - \chi(t)x(t)\right]dt + \sigma_r(t)dW(t) \\
&= \left[f'(0,t) + y(t) - \chi(t)\left(r(t) - f(0,t)\right)\right]dt + \sigma_r(t)dW(t) \\
&= \Bigg[\underbrace{f'(0,t) + \chi(t)f(0,t) + y(t)}_{\theta(t)} - \underbrace{\chi(t)}_{a} r(t)\Bigg] dt + \underbrace{\sigma_r(t)}_{\sigma(t)} dW(t)
\end{aligned}
$$

HJM volatility parameters become

$$
H'(t) = -aH(t), \quad H(0) = 1 \Rightarrow h(t) = H(t) = e^{-at},
$$

$$
g(t) = \sigma_r(t) \cdot H(t)^{-1} = \sigma(t)e^{at}.
$$

Deterministic volatility allows calculation of auxilliary variable $y(t)$

We have

$$y'(t) = \sigma(t)^2 - 2 \cdot a \cdot y(t), \quad y(0) = 0.$$

Solving initial value problem yields

$$y(t) = \int_0^t \sigma(u)^2 \cdot e^{-2a(t-u)} du.$$

Hull-White model in HJM notation

In the HJM framework the Hull-White model becomes

$$
\begin{aligned}
r(t) &= f(0,t) + x(t), \\
dx(t) &= \left[\int_0^t \sigma(u)^2 \cdot e^{-2a(t-u)} du - a \cdot x(t)\right] \cdot dt + \sigma(t) \cdot dW(t), \\
x(0) &= 0.
\end{aligned}
$$

We will use this representation of the Hull-White model for our implementations.

This also gives HJM representation of Hull-White model

Corollary (Forward rate dynamics in Hull-White model)

In a Hull-White model the dynamics of the forward rate $f(t,T)$ become

$$df(t,T) = \sigma(t)^2 e^{-a(T-t)} \frac{1 - e^{-a(T-t)}}{a} dt + \sigma(t) e^{-a(T-t)} dW(t).$$

Proof.

$$\begin{aligned} df(t,T) &= \sigma_f(t,T) \cdot \left[\int_t^T \sigma_f(t,u) du \right] \cdot dt + \sigma_f(t,T) \cdot dW(t) \\ &= g(t)h(T) \left[\int_t^T g(t)h(u) du \right] \cdot dt + g(t)h(T) \cdot dW(t) \\ &= \sigma(t)^2 e^{-a(T-t)} \underbrace{\left[\int_t^T e^{-a(u-t)} du \right]}_{\frac{1-e^{-a(T-t)}}{a}} \cdot dt + \sigma(t) e^{-a(T-t)} \cdot dW(t). \end{aligned}$$

Zero bond prices may also be computed in terms of $x(t)$

Corollary (Zero bonds in Hull-White model)

In the Hull-White model future zero coupon bonds are

$$P(t,T) = \frac{P(0,T)}{P(0,t)} \exp\left\{ -G(t,T)x(t) - \frac{G(t,T)^2}{2} \int_0^t \left[e^{-a(t-u)} \sigma(u) \right]^2 du \right\}$$

with

$$G(t,T) = \int_t^T e^{-a(u-t)} du = \left[\frac{1 - e^{-a(T-t)}}{a} \right].$$

Proof.

Result follows either from Hull-White model zero bond formula with $x(t) = r(t) - f(0,T)$ or from zero bond formula for the separable HJM model with Hull-White results for $G(t,T)$ and $y(t)$. □

Outline

First we need the distribution of the state variable $x(t)$

We have

$$dx(t) = [y(t) - a \cdot x(t)] \cdot dt + \sigma(t) \cdot dW(t).$$

This yields for $t \geq s$

$$x(t) = e^{-a(t-s)} \left[x(s) + \int_s^t e^{a(u-s)} \left(y(u)du + \sigma(u)dW(u) \right) \right].$$

Lemma (State variable distribution)

In the HJM version of the Hull-White model we have that under the risk-neutral measure the state variable $x(t)$ is normally distributed with

$$\mathbb{E}^{\mathbb{Q}} \left[x(t) \,|\, \mathcal{F}_s \right] = e^{-a(t-s)} \left[x(s) + \int_s^t e^{a(u-s)} y(u) du \right] \text{ and}$$

$$Var \left[x(t) \,|\, \mathcal{F}_s \right] = \int_s^t \left[e^{-a(t-u)} \sigma(u) \right]^2 du.$$

Result follows directly from state variable representation for $x(t)$

Proof.

Result for $\mathbb{E}\left[x(t) \,|\, \mathcal{F}_s\right]$ follows from martingale property of Ito integral. Variance follows from Ito isometry

$$\begin{aligned}\mathrm{Var}\left[x(t) \,|\, \mathcal{F}_s\right] &= e^{-2a(t-s)} \int_s^t \left[e^{a(u-s)}\sigma(u)\right]^2 du \\ &= \int_s^t \left[e^{-a(t-u)}\sigma(u)\right]^2 du.\end{aligned}$$

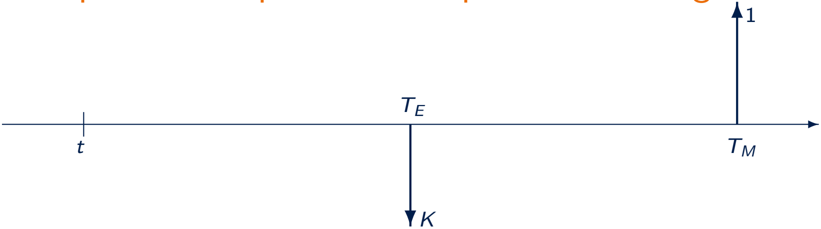
$\square$

- We will have a closer look at $\mathbb{E}^{\mathbb{Q}}\left[x(t) \,|\, \mathcal{F}_s\right] = e^{-a(t-s)}\left[x(s) + \int_s^t e^{a(u-s)} y(u) du\right]$ later on.
- Note, that we can also write

$$\mathrm{Var}\left[x(t) \,|\, \mathcal{F}_s\right] = y(t) - G'(s,t)^2 y(s).$$

Auxilliary variable $y(t)$ represents the (co-)variance process of $x(t)$.

Zero coupon bond options are important building blocks



Definition (Zero coupon bond (ZCB) option)

A zero coupon bond option is defined as an option with expiry time T_E, ZCB maturity time T_M with $T_M \geq T_E$, strike K , call/put flag $\phi \in \{1, -1\}$ and payoff

$$V^{\text{ZBO}}(T_E) = \left[\phi \left(P(T_E, T_M) - K\right)\right]^+ .$$

- We are interested in present value $V^{\text{ZBO}}(t)$.
- We use T_E-forward measure for valuation

$$V^{\text{ZBO}}(t) = P(t, T_E) \cdot \mathbb{E}^{T_E} \left[\left[\phi \left(P(T_E, T_M) - K\right)\right]^+ \mid \mathcal{F}_t \right].$$

$P(T_E, T_M)$ is log-normally distributed with known parameters

- We have for the forward bond price

$$\mathbb{E}^{T_E}\left[P(T_E, T_M) \,|\, \mathcal{F}_t\right] = P(t, T_M)/P(t, T_E).$$

- From

$$P(T_E, T_M) = \frac{P(t, T_M)}{P(t, T_E)} e^{-G(T_E, T_M)x(T_E) - \frac{G(T_E, T_M)^2}{2} \int_t^{T_E} \left[e^{-a(T_E-u)}\sigma(u)\right]^2 du}$$

  we get
  - $P(T_E, T_M)$ is log-normally distributed with log-normal variance

$$\nu^2 = \mathrm{Var}\left[G(T_E, T_M)x(T_E) \,|\, \mathcal{F}_t\right] = G(T_E, T_M)^2 \int_t^{T_E} \left[e^{-a(T_E-u)}\sigma(u)\right]^2 du,$$

  - we can apply Black's formula for option pricing.

ZCO prices are given by Black's formula

Theorem (ZCO pricing formula)

The time-t price of a zero coupon bond option with expiry time T_E, ZCB maturity time T_M with $T_M \geq T_E$, strike K , call/put flag $\phi \in \{1, -1\}$ and payoff

$$V^{ZBO}(T_E) = [\phi (P(T_E, T_M) - K)]^+$$

is given by

$$V^{ZBO}(t) = P(t, T_E) \cdot Black(P(t, T_M)/P(t, T_E), K, \nu, \phi)$$

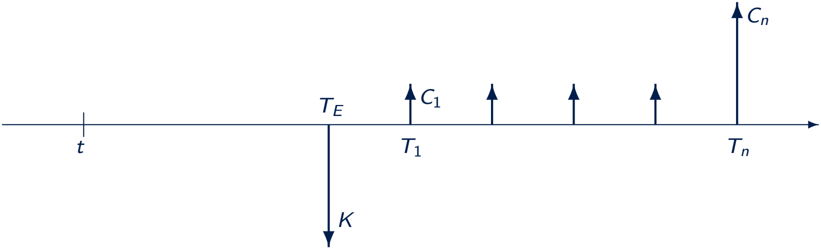
with log-normal bond price variance

$$\nu^2 = G(T_E, T_M)^2 \int_t^{T_E} \left[e^{-a(T_E-u)} \sigma(u) \right]^2 du.$$

Proof.

Result follows from log-normal distribution property. □

Coupon bond options are further building blocks



Payoff at option expiry T_E

$$V(T_E) = \left[\left(\sum_{i=1}^{n} C_i \cdot P(T_E, T_i) \right) - K \right]^+ .$$

Coupon bond options are options on a basket of future cash flows

Definition (Coupon bond option (CBO))

A coupon bond option is defined as an option with expiry time T_E, future cash flow payment times $T_1, \ldots, T_n$ (with $T_i > T_E$), corresponding cash flow values $C_1, \ldots, C_n$, a fixed strike price K, call/put flag $\phi \in \{1, -1\}$ and payoff

$$V^{\text{CBO}}(T_E) = \left[\left(\phi \left[\left(\sum_{i=1}^{n} C_i P(T_E, T_i) \right) - K \right] \right)^+ \right].$$

- We cannot price CBO directly due to the basket structure.
- However, with some (not too strong) assumptions we can represent the 'option on a basket' as a 'basket of options'.
- We use monotonicity of bond prices (for $t < T$)

$$\frac{\partial}{\partial x} P(x(t); t, T) = -G(t, T) \cdot P(x(t); t, T) < 0.$$

CBO's are transformed via Jamshidian's trick I

W.l.o.g. set $\phi = 1$ (method works for $\phi = -1$ as well).
Assume underlying bond is monotone in state variable $x(T_E)$, i.e.

$$\begin{aligned}\frac{\partial}{\partial x}\sum_{i=1}^{n} C_i P(x(T_E); T_E, T_i) &= \sum_{i=1}^{n} C_i \frac{\partial}{\partial x} P(x(T_E); T_E, T_i) \\ &= -\sum_{i=1}^{n} C_i G(T_E, T_i) P(x(T_E); T_E, T_i) < 0.\end{aligned}$$

- Condition is satisfied, e.g. if $C_i \geq 0$.
- Small negative cash flows typically don't violate the assumption since last cash flow C_n is typically a large positive cash flow.

CBO's are transformed via Jamshidian's trick II

Then find $x^\star$ such that

$$\left(\sum_{i=1}^{n} C_i P(x^\star; T_E, T_i)\right) - K = 0$$

and set $K_i = P(x^\star; T_E, T_i)$.
We get (using monotonicity assumption)

$$\begin{aligned}
\left[\left(\sum_{i=1}^{n} C_i P(T_E, T_i)\right) - K\right]^+ &= \mathbb{1}_{\{x(T_E)\le x^\star\}}\left[\left(\sum_{i=1}^{n} C_i P(T_E, T_i)\right) - K\right] \\
&= \mathbb{1}_{\{x(T_E)\le x^\star\}}\left[\sum_{i=1}^{n} C_i P(T_E, T_i) - \sum_{i=1}^{n} C_i K_i\right] \\
&= \left[\sum_{i=1}^{n} C_i \left[P(T_E, T_i) - K_i\right] \mathbb{1}_{\{x(T_E)\le x^\star\}}\right] \\
&= \left[\sum_{i=1}^{n} C_i \left[P(T_E, T_i) - K_i\right]^+\right].
\end{aligned}$$

CBO's are transformed via Jamshidian's trick III

This gives

$$\mathbb{E}^{T_E}\left[\left[\left(\sum_{i=1}^{n} C_i P(T_E, T_i)\right) - K\right]^+\right] = \sum_{i=1}^{n} C_i \underbrace{\mathbb{E}^{T_E}\left[[P(T_E, T_i) - K_i]^+\right]}_{\text{Black's formula}}$$

or

$$\begin{aligned}
V^{\text{CBO}}(t) &= \sum_{i=1}^{n} C_i \cdot V_i^{\text{ZBO}}(t) \\
&= \sum_{i=1}^{n} C_i \cdot P(t, T_E) \cdot \text{Black}\left(P(t, T_i)/P(t, T_E), K_i, \nu_i, \phi\right), \\
\nu_i^2 &= G(T_E, T_i)^2 \int_t^{T_E} \left[e^{-a(T_E - u)}\sigma(u)\right]^2 du.
\end{aligned}$$

CBO's are prices as sum of ZBO's

Theorem (CBO pricing formula)

Consider a CBO with expiry time T_E, future cash flow payment times $T_1, \ldots, T_n$ (with $T_i > T_E$), corresponding cash flow values $C_1, \ldots, C_n$, fixed strike price K and call/put flag $\phi \in \{1, -1\}$. Assume that the underlying bond price $\sum_{i=1}^{n} C_i P(x(T_E); T_E, T_i)$ is monotonically decreasing in the state variable $x(T_E)$. Then the time-t price of the CBO is

$$V^{CBO}(t) = \sum_{i=1}^{n} C_i \cdot V_i^{ZBO}(t)$$

where $V_i^{ZBO}(t)$ is the time-t price of a corresponding ZBO with strike $K_i = P(x^\star; T_E, T_i)$ where the break-even state $x^\star$ is given by

$$\left(\sum_{i=1}^{n} C_i P(x^\star; T_E, T_i) \right) - K = 0.$$

Proof.

Follows from derivation above. $\square$

Outline

We have another look at the expectation(s) of $x(t)$

- For general option pricing we also need expectation $\mathbb{E}^T\left[x(T) \mid \mathcal{F}_t\right]$.
- Then we can price

$$V(t) = P(t,T) \cdot \mathbb{E}^T\left[V(x(T);T) \mid \mathcal{F}_t\right] = P(t,T) \cdot \int_{-\infty}^{+\infty} V(x;T) \cdot p_{\mu,\sigma^2}(x) \cdot dx.$$

- Here $p_{\mu,\sigma^2}(x)$ is the density of a normal distribution $N\left(\mu,\sigma^2\right)$ with

$$\mu = \mathbb{E}^T\left[x(T) \mid \mathcal{F}_t\right] \text{ and } \sigma^2 = \mathrm{Var}\left[x(T) \mid \mathcal{F}_t\right].$$

- Integral $\int_{-\infty}^{+\infty} V(x;T) \cdot p_{\mu,\sigma^2}(x) \cdot dx$ is typically evaluated numerically (i.e. quadrature).
- We first calculate $\mathbb{E}^{\mathbb{Q}}\left[x(T) \mid \mathcal{F}_t\right]$ and then derive $\mathbb{E}^T\left[x(T) \mid \mathcal{F}_t\right]$.

We calculate expectation in risk-neutral measure I

Recall

$$dx(t) = [y(t) - a \cdot x(t)] \cdot dt + \sigma(t) \cdot dW(t).$$

This yields for $T \geq t$

$$x(T) = e^{-a(T-t)} \left[x(t) + \int_t^T e^{a(u-t)} \left(y(u)du + \sigma(u)dW(u) \right) \right]$$

and

$$\mathbb{E}^{\mathbb{Q}} \left[x(T) \,|\, \mathcal{F}_t \right] = e^{-a(T-t)} x(t) + \int_t^T e^{-a(T-u)} y(u) du.$$

We get

$$\begin{aligned}
\int_t^T e^{-a(T-u)} y(u) du &= \int_t^T e^{-a(T-u)} \left(\int_0^u \sigma(s)^2 e^{-2a(u-s)} ds \right) du \\
&= \int_t^T e^{-a(T-u)} \left(\int_0^t \sigma(s)^2 e^{-2a(u-s)} ds \right) du \\
&\quad + \int_t^T e^{-a(T-u)} \left(\int_t^u \sigma(s)^2 e^{-2a(u-s)} ds \right) du.
\end{aligned}$$

We calculate expectation in risk-neutral measure II

We analyse the integrals individually,

$$
\begin{aligned}
I_1(t,T) &= \int_t^T e^{-a(T-u)}\left(\int_0^t \sigma(s)^2 e^{-2a(u-s)}ds\right)du\\
&= \int_t^T \left(\int_0^t e^{-a(T-u)}\sigma(s)^2 e^{-2a(u-s)}ds\right)du\\
&= \int_0^t \left(\int_t^T e^{-a(T-u)}\sigma(s)^2 e^{-2a(u-s)}du\right)ds\\
&= \int_0^t \sigma(s)^2\left(\int_t^T e^{-a(T-u)}e^{-2a(u-s)}du\right)ds\\
&= \int_0^t \sigma(s)^2\left[\frac{e^{-a(T-u)}e^{-2a(u-s)}}{-a}\right]_{u=t}^{T}ds\\
&= \int_0^t \frac{\sigma(s)^2}{a}\left[e^{-a(T-t)}e^{-2a(t-s)} - e^{-a(T-T)}e^{-2a(T-s)}\right]ds.
\end{aligned}
$$

We calculate expectation in risk-neutral measure III

Exponential terms can be further simplified as

$$e^{-a(T-t)}e^{-2a(t-s)} - e^{-2a(T-s)} = e^{-a(T-t)}\left[1 - e^{-a(T-t)}\right]e^{-2a(t-s)}.$$

This gives

$$I_1(t,T) = e^{-a(T-t)}\frac{1 - e^{-a(T-t)}}{a}\int_0^t \sigma(s)^2 e^{-2a(t-s)}ds.$$

We calculate expectation in risk-neutral measure IV

For the second integral we get

$$
\begin{aligned}
I_2(t,T) &= \int_t^T e^{-a(T-u)} \left(\int_t^u \sigma(s)^2 e^{-2a(u-s)} ds \right) du \\
&= \int_t^T \left(\int_t^u e^{-a(T-u)} \sigma(s)^2 e^{-2a(u-s)} ds \right) du \\
&= \int_t^T \left(\int_s^T e^{-a(T-u)} \sigma(s)^2 e^{-2a(u-s)} du \right) ds \\
&= \int_t^T \sigma(s)^2 \left(\int_s^T e^{-a(T-u)} e^{-2a(u-s)} du \right) ds \\
&= \int_t^T \sigma(s)^2 \left[\frac{e^{-a(T-u)} e^{-2a(u-s)}}{-a} \right]_{u=s}^T ds \\
&= \int_t^T \frac{\sigma(s)^2}{a} \left[e^{-a(T-s)} e^{-2a(s-s)} - e^{-a(T-T)} e^{-2a(T-s)} \right] ds.
\end{aligned}
$$

We calculate expectation in risk-neutral measure V

Again we simplify exponential terms

$$e^{-a(T-s)} - e^{-2a(T-s)} = e^{-a(T-s)}\left[1 - e^{-a(T-s)}\right].$$

This gives

$$I_2(t,T) = \int_t^T \sigma(s)^2 e^{-a(T-s)} \frac{1 - e^{-a(T-s)}}{a} ds.$$

In summary, we get

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}\left[x(T) \,|\, \mathcal{F}_t\right] &= e^{-a(T-t)} x(t) + I_1(t,T) + I_2(t,T) \\
&= e^{-a(T-t)}\left[x(t) + \frac{1 - e^{-a(T-t)}}{a} \int_0^t \sigma(s)^2 e^{-2a(t-s)} ds\right] \\
&\quad + \int_t^T \sigma(s)^2 e^{-a(T-s)} \frac{1 - e^{-a(T-s)}}{a} ds.
\end{aligned}$$

We calculate expectation in terminal measure I

Recall change of measure

$$dW^T(t) = dW(t) + \sigma_P(t,T)dt.$$

We have

$$\sigma_P(t,T) = \sigma(t)G(t,T) = \sigma(t)\cdot\frac{1-e^{-a(T-t)}}{a}.$$

This gives

$$dx(t) = \left[y(t) - \sigma(t)^2 G(t,T) - a\cdot x(t)\right]\cdot dt + \sigma(t)\cdot dW^T(t)$$

and

$$x(T) = e^{-a(T-t)}\cdot \left[x(t) + \int_t^T e^{a(u-t)}\left(\left[y(u) - \sigma(u)^2 G(u,T)\right]du + \sigma(u)dW^T(u)\right)\right].$$

We calculate expectation in terminal measure II

We find that

$$\mathbb{E}^{T}\left[x(T)\,|\,\mathcal{F}_t\right] = \mathbb{E}^{\mathbb{Q}}\left[x(T)\,|\,\mathcal{F}_t\right] - \int_t^T \sigma(u)^2 e^{-a(T-u)} G(u,T)du.$$

It turns out that

$$\begin{aligned}\int_t^T \sigma(u)^2 e^{-a(T-u)} G(u,T)du &= \int_t^T \sigma(u)^2 e^{-a(T-u)} \frac{1-e^{-a(T-u)}}{a} du \\ &= I_2(t,T).\end{aligned}$$

As a result, we get

$$\mathbb{E}^{T}\left[x(T)\,|\,\mathcal{F}_t\right] = e^{-a(T-t)}\left[x(t) + \frac{1-e^{-a(T-t)}}{a}\int_0^t \sigma(s)^2 e^{-2a(t-s)} ds\right]$$

or more formally

$$\mathbb{E}^{T}\left[x(T)\,|\,\mathcal{F}_t\right] = G'(t,T)\left[x(t) + G(t,T)y(t)\right].$$

Outline

All the formulas serve the purpose of model calibration and derivative pricing

Model Calibration	Derivative Pricing
zero bond option (ZBO)	future zero bonds $P(x(t); t, T)$
coupon bond option (CBO)	expectation $\mathbb{E}^T[x(T) \mid \mathcal{F}_t]$ and variance $\mathrm{Var}[x(T) \mid \mathcal{F}_t]$
European swaption	payoff pricing $V(t) = P(t,T) \cdot \mathbb{E}^T[V(x(T); T) \mid \mathcal{F}_t]$

Bond option pricing is realised via ZBO's and CBO's

Zero Bond Option (ZBO)

Zero bond with expiry T_E, maturity T_M, strike K and call/put flag ϕ

$$\begin{aligned} V^{\text{ZBO}}(0) &= P(0, T_E) \cdot \text{Black}\left(P(0, T_M)/P(0, T_E), K, \nu, \phi\right), \\ \nu^2 &= G(T_E, T_M)^2 y(T_E). \end{aligned}$$

Coupon Bond Option (CBO)

Coupon bond option with strike K and underlying bond $\sum_{i=1}^{n} C_i \cdot P(T_E, T_i)$,

$$V^{\text{CBO}}(t) = \sum_{i=1}^{n} C_i \cdot V_i^{\text{ZBO}}(t)$$

where ZBO's $V_i^{\text{ZBO}}(t)$ with expiry T_E, maturity T_i, and strike $K_i = P(x^\star, T_E, T_i)$ and $x^\star$ s.t.

$$\sum_{i=1}^{n} C_i \cdot P(x^\star; T_E, T_i) = K.$$

General derivative pricing requires state variable expectation and variance

Zero Bonds (as building blocks for payoffs $V(x(T);T)$)

$$P(x(T);T,S) = \frac{P(0,S)}{P(0,T)} \exp\left\{-G(T,S)x(T) - \frac{G(T,S)^2}{2}y(T)\right\}.$$

General Derivative Pricing

$$V(t) = P(t,T)\cdot\mathbb{E}^T\left[V(x(T);T)\,|\,\mathcal{F}_t\right] = P(t,T)\cdot\int_{-\infty}^{+\infty} V(x;T)\cdot p_{\mu,\sigma^2}(x)\cdot dx$$

with $p_{\mu,\sigma^2}(x)$ density of a Normal distribution $N\left(\mu,\sigma^2\right)$ with

$$\mu = \mathbb{E}^T\left[x(T)\,|\,\mathcal{F}_t\right] = G'(t,T)\left[x(t) + G(t,T)y(t)\right]$$

and

$$\sigma^2 = \mathrm{Var}\left[x(T)\,|\,\mathcal{F}_t\right] = y(T) - G'(t,T)^2 y(t).$$

Fortunately, we only need a small set of model functions for implementation

- Discount factors $P(0, T)$ from input yield curve.
- Function $G(t, T)$ with

$$G(t, T) = \frac{1 - e^{-a(T-t)}}{a}.$$

- Function $G'(t, T)$ with

$$G'(t, T) = e^{-a(T-t)}.$$

- Auxilliary variable $y(t)$ with

$$y(t) = \int_0^t \left[e^{-a(t-u)} \sigma(u) \right]^2 du = \sum_{j=1}^{k} \frac{e^{-2a(t-t_j)} - e^{-2a(t-t_{j-1})}}{2a} \sigma_j^2$$

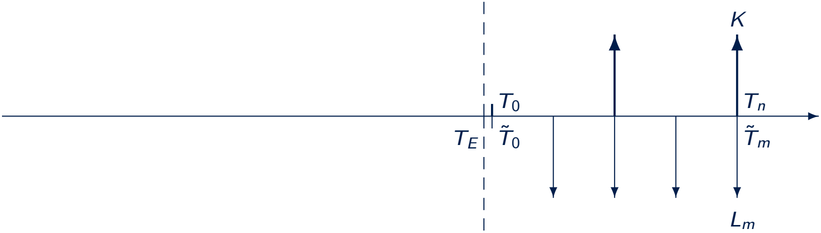
where we assume $\sigma(t)$ piece-wise constant on a grid $0 = t_0, t_1, \ldots, t_k = t$.

Outline

It remains to show how Hull-Wite model is applied to European swaptions

Model Calibration	Derivative Pricing
zero bond option (ZBO)	future zero bonds $P(x(t); t, T)$
coupon bond option (CBO)	expectation $\mathbb{E}^T\left[x(T) \mid \mathcal{F}_t\right]$ and variance $\mathrm{Var}\left[x(T) \mid \mathcal{F}_t\right]$
European swaption	payoff pricing $V(t) = P(t, T) \cdot \mathbb{E}^T\left[V(x(T); T) \mid \mathcal{F}_t\right]$

Recall that Swaption is option to enter into a swap at a future time



- At option exercise time T_E present value of swap is

$$V^{\text{Swap}}(T_E) = \underbrace{K \sum_{i=1}^{n} \tau_i P(T_E, T_i)}_{\text{future fixed leg}} - \underbrace{\sum_{j=1}^{m} L^{\delta}(T_E, \tilde{T}_{j-1}, \tilde{T}_{j-1} + \delta)\tilde{\tau}_j P(T_E, \tilde{T}_j)}_{\text{future float leg}}.$$

- Option to enter represents the right but not the obligation to enter swap.
- Rational market participant will exercise if swap present value is positive, i.e.

$$V^{\text{Swpt}}(T_E) = \max\left\{V^{\text{Swap}}(T_E), 0\right\}.$$

How do we get the swaption payoff compatible to our Hull-White model formulas?

$$V^{\mathrm{Swap}}(T_E) = \underbrace{K \sum_{i=1}^{n} \tau_i P(T_E, T_i)}_{\text{future fixed Leg}} - \underbrace{\sum_{j=1}^{m} L^{\delta}(T_E, \tilde{T}_{j-1}, \tilde{T}_{j-1} + \delta)\tilde{\tau}_j P(T_E, \tilde{T}_j)}_{\text{future float leg}}$$

- Fixed leg can be expressed in terms of future state variable $x(T_E)$ via $P(x(T_E); T_E, T_i)$
- Float leg contains future forward Libor rates $L^{\delta}(T_E, \tilde{T}_{j-1}, \tilde{T}_{j-1} + \delta)$ from (future) projection curve
- However, Hull-White model only provides representation of discount factors, i.e. $P(T_E, \tilde{T}_j)$

We need to model the relation between future Libor rates $L^{\delta}(T_E, \tilde{T}_{j-1}, \tilde{T}_{j-1} + \delta)$ and discount factors $P(T_E, \tilde{T}_j)$.

We do have all ingredients from our deterministic multi-curve model

Recall the definition of (future) forward Libor rate

$$
\begin{aligned}
L^{\delta}(T_E, \tilde{T}_{j-1}, \tilde{T}_{j-1}+\delta) &= \mathbb{E}^{\tilde{T}_{j-1}+\delta}\left[L^{\delta}(\tilde{T}_{j-1}, \tilde{T}_{j-1}, \tilde{T}_{j-1}+\delta) \mid \mathcal{F}_{T_E}\right] \\
&= \left[\frac{P(T_E, \tilde{T}_{j-1})}{P(T_E, \tilde{T}_{j-1}+\delta)} \cdot D(\tilde{T}_{j-1}, \tilde{T}_{j-1}+\delta) - 1\right]\frac{1}{\tau_{j-1}}
\end{aligned}
$$

($\tau_{j-1} = \tau(\tilde{T}_{j-1}, \tilde{T}_{j-1}+\delta)$) with tenor basis factor

$$
D(\tilde{T}_{j-1}, \tilde{T}_{j-1}+\delta) = \frac{Q(T_E, \tilde{T}_{j-1})}{Q(T_E, \tilde{T}_{j-1}+\delta)}
$$

and discount factors $Q(T_E, T)$ arising from credit (or funding) risk embedded in Libor rates $L^{\delta}(\cdot)$.

- Key assumption is that $D(\tilde{T}_{j-1}, \tilde{T}_{j-1}+\delta)$ is deterministic or independent of T_E.
- Then

$$
D(\tilde{T}_{j-1}, \tilde{T}_{j-1}+\delta) = \frac{Q(0, \tilde{T}_{j-1})}{Q(0, \tilde{T}_{j-1}+\delta)} = \frac{P^{\delta}(0, \tilde{T}_{j-1})}{P^{\delta}(0, \tilde{T}_{j-1}+\delta)} \cdot \frac{P(0, \tilde{T}_{j-1}+\delta)}{P(0, \tilde{T}_{j-1})}.
$$

We use basis spread model to simplify Libor coupons

- Tenor basis factor

$$D_{j-1} = D(\tilde{T}_{j-1}, \tilde{T}_{j-1} + \delta) = \frac{P^{\delta}(0, \tilde{T}_{j-1})}{P^{\delta}(0, \tilde{T}_{j-1} + \delta)} \cdot \frac{P(0, \tilde{T}_{j-1} + \delta)}{P(0, \tilde{T}_{j-1})}$$

  is calculated from today's projection curve $P^{\delta}(0, T)$ and discount curve $P(0, T)$.

- Further assume *natural* Libor payment dates and consistent year fractions

$$\tilde{T}_j = \tilde{T}_{j-1} + \delta, \quad \tau(\tilde{T}_{j-1}, \tilde{T}_{j-1} + \delta) = \tilde{\tau}_j.$$

- Libor coupon becomes

$$\begin{aligned} L^{\delta}(T_E, \tilde{T}_{j-1}, \tilde{T}_j)\tilde{\tau}_j P(T_E, \tilde{T}_j) &= \left[\frac{P(T_E, \tilde{T}_{j-1})}{P(T_E, \tilde{T}_j)} D_{j-1} - 1 \right] \frac{1}{\tilde{\tau}_j} \tilde{\tau}_j P(T_E, \tilde{T}_j) \\ &= P(T_E, \tilde{T}_{j-1}) D_{j-1} - P(T_E, \tilde{T}_j). \end{aligned}$$

We can write the float leg (1/2)

$$
\begin{aligned}
V^{\text{Swap}}(T_E) &= \underbrace{K\sum_{i=1}^{n}\tau_i P(T_E,T_i)}_{\text{future fixed leg}} - \underbrace{\sum_{j=1}^{m} L^{\delta}(T_E,\tilde{T}_{j-1},\tilde{T}_{j-1}+\delta)\tilde{\tau}_j P(T_E,\tilde{T}_j)}_{\text{future float leg}} \\
&= K\sum_{i=l}^{n}\tau_i P(T_E,T_i) - \sum_{j=1}^{m} P(T_E,\tilde{T}_{j-1})D_{j-1} - P(T_E,\tilde{T}_j) \\
&= K\sum_{i=1}^{n}\tau_i P(T_E,T_i) \\
&\quad - \left[P(T_E,\tilde{T}_0)D_0 - P(T_E,\tilde{T}_m) + \sum_{j=2}^{m} P(T_E,\tilde{T}_{j-1})\left[D_{j-1}-1\right]\right] \\
&= K\sum_{i=1}^{n}\tau_i P(T_E,T_i) \\
&\quad - \left[P(T_E,\tilde{T}_0) - P(T_E,\tilde{T}_m) + \sum_{j=1}^{m} P(T_E,\tilde{T}_{j-1})\left[D_{j-1}-1\right]\right].
\end{aligned}
$$

We can re-write the float leg (2/2)

Reordering terms yields

$$V^{\mathrm{Swap}}(T_E) = -\underbrace{P(T_E,\tilde{T}_0)}_{\text{strike paid at } T_0} + \underbrace{\sum_{i=1}^{n} K\cdot\tau_i\cdot P(T_E,T_i)}_{\text{fixed rate coupons}}$$

$$\underbrace{-\sum_{j=1}^{m} P(T_E,\tilde{T}_{j-1})\cdot\left[D_{j-1}-1\right]}_{\text{negative spread coupons}} + \underbrace{P(T_E,\tilde{T}_m)}_{\text{notional payment}}$$

$$= \sum_{k=0}^{n+m+1} C_k\cdot P(T_E,\bar{T}_k)$$

with

$$C_0=-1,\ C_i=K\cdot\tau_i\,(i=1,\dots,n),\ C_{n+j}=-\left[D_{j-1}-1\right],\ (j=1,\dots,m),$$

$$\text{and } C_{n+m+1}=1,$$

and corresponding payment times $\bar{T}_k$.

Swaptions are equivalent to coupon bond options

Corollary (Equivalence between Swaption and bond option)

Consider a European Swaption with receiver/payer flag $\phi \in \{1, -1\}$ payoff

$$V^{Swpt}(T_E) = \Bigg[\phi \Bigg\{ K \sum_{i=1}^{n} \tau_i P(T_E, T_i) - \sum_{j=1}^{m} L^{\delta}(T_E, \tilde{T}_{j-1}, \tilde{T}_{j-1} + \delta) \tilde{\tau}_j P(T_E, \tilde{T}_j) \Bigg\}$$

Under our deterministic basis spread assumption the swaption payoff is equal to a call/put bond option payoff

$$V^{CBO}(T_E) = \left[\phi \left\{ \sum_{k=0}^{n+m+1} C_k \cdot P(T_E, \bar{T}_k) \right\} \right]^+$$

with zero strike and cash flows C_k and times $\bar{T}_k$ as elaborated above. Moreover, if the underlying bond payoff is monotonic then

$$V^{Swpt}(t) = V^{CBO}(t) = \sum_{k=0}^{n+m+1} C_k \cdot V_k^{ZBO}(t)$$

We give some comments regarding the CBO mapping

- Note that $C_0 = -1$ is a *large* negative cash flow.
- However, $\frac{\partial}{\partial x}\left[-P(T_E, \tilde{T}_0)\right] \approx -G(T_E, T_0)$ is small because $T_E - T_0$ is small.
- If $T_E = \tilde{T}_0$, i.e. no spot offset between option expiry and swap start time, then
  - set CBO strike $K = D(\tilde{T}_0, \tilde{T}_1)$,
  - remove first negative spread coupon C_{n+1} from cash flow list.
- In practice monotonicity assumption

$$\frac{\partial}{\partial x}\left[\sum_{k=0}^{n+m+1} C_k \cdot P(T_E, \bar{T}_k)\right] < 0$$

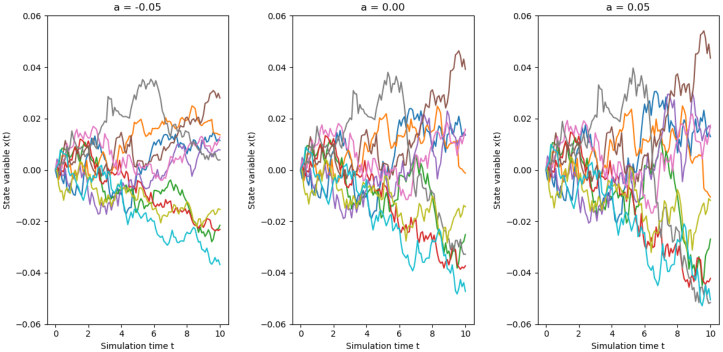
is typically no issue.

In Hull-White model calibration we will use CBO formula to match Hull-White model prices versus Vanilla model swaption prices.

Outline

How do the simulated paths *look like*?

- Model short rate volatility σ calibrated to $100bp$ flat volatility at $5y$ and $10y$, mean reversion $a \in \{-5\%, 0\%, 5\%\}$ [6]



- Higher mean reversion yields more *forward volatility.*

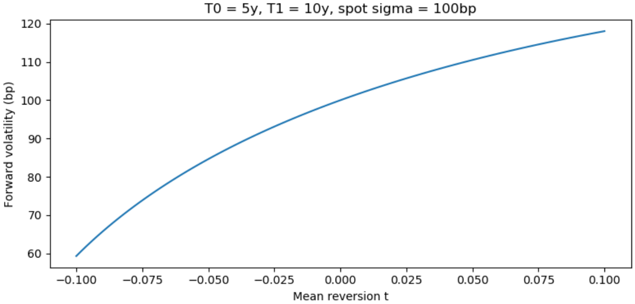
[6] Zero mean reversion is effectively approximated via $a = 1bp$. This does not change the overall behavior and avoids special treatment in formulas.

Forward volatility dependence on mean reversion can also be derived analytically

Denote forward volatility as

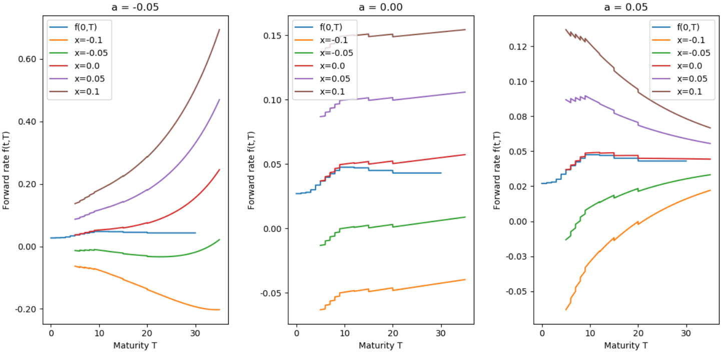
$$\sigma_{\text{Fwd}}(T_0, T_1) = \sqrt{\frac{\text{Var}\left[x(T_1) \mid \mathcal{F}_{T_0}\right]}{T_1 - T_0}} = \sqrt{\frac{y(T_1) - G'(T_0, T_1)^2 y(T_0)}{T_1 - T_0}}$$

- Suppose spot volatilities $\sigma_{\text{Fwd}}(0, T_1)$ and $\sigma_{\text{Fwd}}(0, T_0)$ (and thus $y(T_0)$ and $y(T_1)$ are fixed)
- If mean reversion a increases then $G'(T_0, T_1) = e^{-a(T_1 - T_0)}$ decreases
- Thus forward volatility $\sigma_{\text{Fwd}}(T_0, T_1)$ increases



Which kind of curves can we simulate with Hull-White model?

- Models use flat short rate volatility $\sigma = 100bp$ and mean reversion $a \in \{-5\%, 0\%, 5\%\}$ [7]



- Model *works* with negative mean reversion - however, yield curves are exploding

[7] Zero mean reversion is effectively approximated via $a = 1bp$. This does not change the overall behavior and avoids special treatment in formulas.

What are relevant properties of a model for option pricing?

- Vanilla models require input (ATM volatility) parameters for expiry-tenor-pairs.
  - Which shape of ATM volatilities for expiry-tenor-pairs are predicted by Hull-White model?
- SABR model allows modelling of volatility smile.
  - Which shapes of volatility smile can be modelled with Hull-White model?
  - How does the smile change if we change the model parameters?
- We aim at applying the Hull-White model to price Bermudan swaptions.
  - How do the model parameters impact prices of exotic derivatives?

For now we focus on model-implied volatilities (ATM and smile). The impact of model parameters on Bermudans is analysed later.

Model properties for option pricing are assessed by analysing model-implied volatilities

Model-implied normal volatility

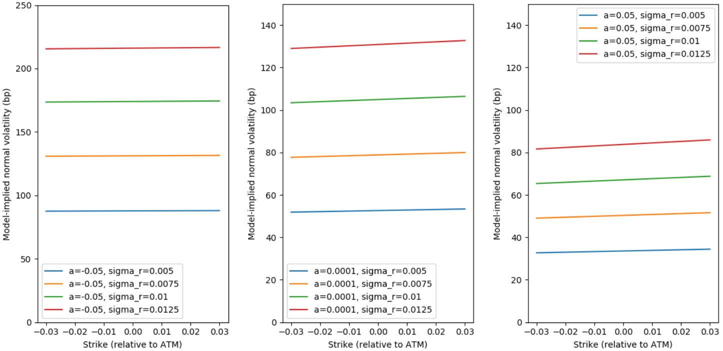
Consider a swaption with expiry/start/end-dates $T_E / T_0 / T_n$ and strike rate K. For a given Hull-White model the model-implied normal volatility is calculated as

$$\sigma(T_0, T_n, K) = \text{Bachelier}^{-1}\left(S(t), K, V^{\text{CBO}}(t)/An(t), \phi\right)/\sqrt{T_E - t}.$$

Here, $S(t)$ and $An(t)$ are the forward swap rate and annuity of the underlying swap with start/end-date T_0 / T_n. $V^{\text{CBO}}(t)$ is the Hull-White model price of a coupon bond option equivalent to the input swaption.

Which shapes of volatility smile can be modelled and how does the smile change if we change the model parameters?

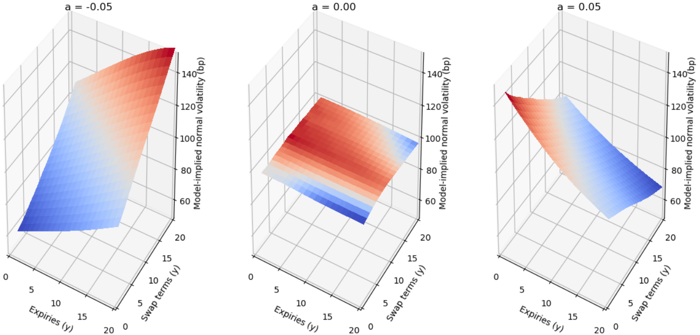
- Models use flat short rate volatility $\sigma \in \{50bp, 75bp, 100bp, 125bp\}$ and mean reversion $a \in \{-5\%, 0\%, 5\%\}$:



- We can only model flat smile - this is a major model limitation!
- Model-implied volatility decreases if mean reversion increases.

Which shape of ATM volatilities for expiry-tenor-pairs are predicted by Hull-White model?

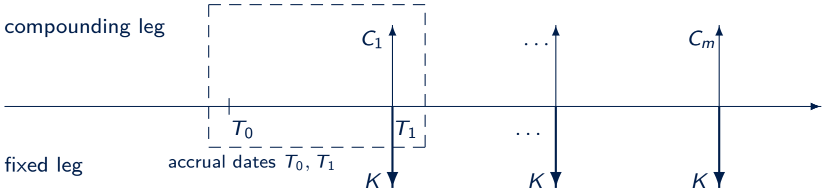
- Models use flat short rate volatility σ - calibrated to 10y-10y swaption with $100bp$ volatility
- Mean reversion $a \in \{-5\%, 0\%, 5\%\}$:

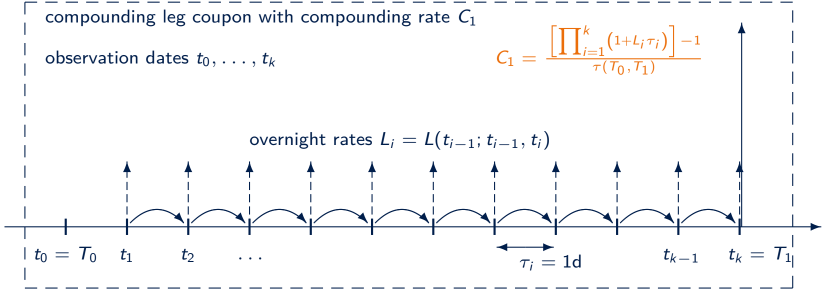


- Mean reversion impacts slope of ATM volatilities in expiry and swap term dimension.

Outline

Recall overnight index swap (OIS) coupon rate calculation





The backward-looking compounded rate is composed of individual overnight rates

- Assume overnight index rate $L_i = L(t_{i-1}; t_{i-1}, t_i)$ is a credit-risk free simple compounded rate.
- Compounded rate C_1 (for a period $[T_0, T_1]$) is payed at T_1 and specified as

$$C_1 = \left\{ \left[\prod_{i=1}^{k} (1 + L_i \tau_i) \right] - 1 \right\} \frac{1}{\tau(T_0, T_1)}.$$

- Crucial part from modeling perspective is compounding factor

$$\prod_{i=1}^{k} (1 + L_i \tau_i) = \prod_{i=1}^{k} \frac{1}{P(t_{i-1}, t_i)}.$$

- Tower-law yields

$$\mathbb{E}^{T_1} \left[\prod_{i=1}^{k} \frac{1}{P(t_{i-1}, t_i)} \,|\, \mathcal{F}_{T_0} \right] = \frac{1}{P(T_0, T_1)}.$$

Outline

For pricing options on compounded rates we need the terminal distribution of the compounding factor

Use Hull-White model representation of zero bonds

$$P(t_{i-1}, t_i) = \frac{P(t, t_i)}{P(t, t_{i-1})} \exp\left\{-G(t_{i-1}, t_i)x(t_{i-1}) - \frac{1}{2}G(t_{i-1}, t_i)^2 y(t_{i-1})\right\},$$

$$G(t_{i-1}, t_i) = \frac{1 - \exp\{-a(t_i - t_{i-1})\}}{a},$$

$$y(t_{i-1}) = \int_t^{t_{i-1}} \sigma(u)^2 \cdot e^{-2a(t_{i-1}-u)} du.$$

Compounding factor becomes

$$\prod_{i=1}^{k} \frac{1}{P(t_{i-1}, t_i)} = \frac{P(t, T_0)}{P(t, T_1)} \exp\left\{\sum_{i=1}^{k} G(t_{i-1}, t_i)x(t_{i-1}) + \frac{1}{2}G(t_{i-1}, t_i)^2 y(t_{i-1})\right\}.$$

Variance of compounding factor is driven by stochastic term $\sum_{i=1}^{k} G(t_{i-1}, t_i)x(t_{i-1})$.

We write all $x(t_{i-1})$ in terms of $x(T_0)$ plus individual Ito integrals

We have in Hull-White model and risk-neutral measure

$$x(t_{i-1}) = e^{-a(t_{i-1}-T_0)}\left[x(T_0) + \int_{T_0}^{t_{i-1}} e^{a(u-T_0)}\left[y(u)du + \sigma(u)dW(u)\right]\right].$$

Abbreviate $dp(u) = y(u)du + \sigma(u)dW(u)$ (to simplify notation). Then

$$\begin{aligned}
&\sum_{i=1}^{k} G(t_{i-1}, t_i)x(t_{i-1}) \\
&= \sum_{i=1}^{k} G(t_{i-1}, t_i)\left\{e^{-a(t_{i-1}-T_0)}\left[x(T_0) + \int_{T_0}^{t_{i-1}} e^{a(u-T_0)}dp(u)\right]\right\} \\
&= x(T_0)\sum_{i=1}^{k} G(t_{i-1}, t_i)e^{-a(t_{i-1}-T_0)} \\
&\qquad + \sum_{i=1}^{k} G(t_{i-1}, t_i)\int_{T_0}^{t_{i-1}} e^{-a(t_{i-1}-u)}dp(u).
\end{aligned}$$

We analyse above two parts individually.

First we calculate the scaling factor for $x(T_0)$

We have

$$G(t_{i-1}, t_i)e^{-a(t_{i-1}-T_0)} = \frac{1-e^{-a(t_i-t_{i-1})}}{a}e^{-a(t_{i-1}-T_0)} = G(T_0, t_i) - G(T_0, t_{i-1}).$$

This yields the telescopic sum

$$\sum_{i=1}^{k} G(t_{i-1}, t_i)e^{-a(t_{i-1}-T_0)} = \sum_{i=1}^{k} G(T_0, t_i) - G(T_0, t_{i-1}) = G(T_0, T_1).$$

And we have

$$x(T_0)\sum_{i=1}^{k} G(t_{i-1}, t_i)e^{-a(t_{i-1}-T_0)} = G(T_0, T_1)x(T_0).$$

Second we calculate the sum of Ito integrals (1/2)

We split integration and re-order sums

$$
\begin{aligned}
&\sum_{i=1}^{k} G(t_{i-1}, t_i) \int_{T_0}^{t_{i-1}} e^{-a(t_{i-1}-u)} dp(u) \\
&= \sum_{i=1}^{k} G(t_{i-1}, t_i) \sum_{j=1}^{i-1} \int_{t_{j-1}}^{t_j} e^{-a(t_{i-1}-u)} dp(u) \\
&= \sum_{i=1}^{k} \sum_{j=1}^{i-1} \int_{t_{j-1}}^{t_j} G(t_{i-1}, t_i) e^{-a(t_{i-1}-u)} dp(u) \\
&= \sum_{i=1}^{k} \sum_{j=1}^{i-1} \int_{t_{j-1}}^{t_j} \left[G(u, t_i) - G(u, t_{i-1}) \right] dp(u) \\
&= \sum_{j=1}^{k-1} \sum_{i=j+1}^{n} \int_{t_{j-1}}^{t_j} \left[G(u, t_i) - G(u, t_{i-1}) \right] dp(u) \\
&= \sum_{j=1}^{k-1} \int_{t_{j-1}}^{t_j} \sum_{i=j+1}^{n} \left[G(u, t_i) - G(u, t_{i-1}) \right] dp(u).
\end{aligned}
$$

Second we calculate the sum of Ito integrals (2/2)

Now we can use telescopic sum property again and simplify

$$
\begin{aligned}
&\sum_{i=1}^{k} G(t_{i-1}, t_i) \int_{T_0}^{t_{i-1}} e^{-a(t_{i-1}-u)} dp(u) \\
&= \sum_{j=1}^{k-1} \int_{t_{j-1}}^{t_j} \sum_{i=j+1}^{n} \left[G(u, t_i) - G(u, t_{i-1}) \right] dp(u) \\
&= \sum_{j=1}^{k-1} \int_{t_{j-1}}^{t_j} \left[G(u, t_n) - G(u, t_j) \right] dp(u) \\
&= \sum_{j=1}^{k-1} G(t_j, t_n) \int_{t_{j-1}}^{t_j} e^{-a(t_j-u)} dp(u).
\end{aligned}
$$

Putting things together yields the desired representation of the compounding factor (1/3)

$$\prod_{i=1}^{k} \frac{1}{P(t_{i-1}, t_i)} = \frac{P(t, T_0)}{P(t, T_1)} \exp\left\{ \sum_{i=1}^{k} G(t_{i-1}, t_i) x(t_{i-1}) + \frac{1}{2} G(t_{i-1}, t_i)^2 y(t_{i-1}) \right\}$$

with

$$\sum_{i=1}^{k} G(t_{i-1}, t_i) x(t_{i-1}) = G(T_0, T_1) x(T_0) + \sum_{j=1}^{k-1} G(t_j, t_n) \int_{t_{j-1}}^{t_j} e^{-a(t_j - u)} dp(u).$$

Putting things together yields the desired representation of the compounding factor (2/3)

Substituting back $dp(u) = y(u)du + \sigma(u)dW(u)$ gives

$$\begin{aligned}\sum_{i=1}^{k} G(t_{i-1}, t_i)x(t_{i-1}) = & \underbrace{G(T_0, T_1)x(T_0)}_{I_0} \\ & + \sum_{j=1}^{k-1} \underbrace{G(t_j, t_n) \int_{t_{j-1}}^{t_j} e^{-a(t_j-u)}\sigma(u)dW(u)}_{I_j} \\ & + \sum_{j=1}^{k-1} G(t_j, t_n) \int_{t_{j-1}}^{t_j} e^{-a(t_j-u)}y(u)du.\end{aligned}$$

Putting things together yields the desired representation of the compounding factor (3/3)

$$\prod_{i=1}^{k} \frac{1}{P(t_{i-1}, t_i)} = \frac{P(t, T_0)}{P(t, T_1)} \exp\left\{ \sum_{i=1}^{k} G(t_{i-1}, t_i) x(t_{i-1}) + \frac{1}{2} G(t_{i-1}, t_i)^2 y(t_{i-1}) \right\}$$

with

$$\begin{aligned}\sum_{i=1}^{k} G(t_{i-1}, t_i) x(t_{i-1}) = & \underbrace{G(T_0, T_1) x(T_0)}_{I_0} \\ & + \sum_{j=1}^{k-1} \underbrace{G(t_j, t_n) \int_{t_{j-1}}^{t_j} e^{-a(t_j - u)} \sigma(u) dW(u)}_{I_j} \\ & + \sum_{j=1}^{k-1} G(t_j, t_n) \int_{t_{j-1}}^{t_j} e^{-a(t_j - u)} y(u) du.\end{aligned}$$

Stochastic Terms I_0 and I_j are independent Ito integrals. Thus $\prod_{i=1}^{k} \frac{1}{P(t_{i-1}, t_i)}$ is log-normal with known variance.

Log-normal variance is given by sum of variances for Ito integrals I_0 and I_j

We first calculate the variance

$$\begin{aligned}
\nu^2 &= \mathrm{Var}\left[\log\left(\prod_{i=1}^{k}\frac{1}{P(t_{i-1},t_i)}\right) \mid \mathcal{F}_t\right] = \mathrm{Var}\left[I_0 + \sum_{j=1}^{k-1} I_j \mid \mathcal{F}_t\right] \\
&= G(T_0,T_1)^2\mathrm{Var}\left[x(T_0) \mid \mathcal{F}_t\right] \\
&\quad + \sum_{j=1}^{k-1} \mathbb{1}_{\{t\le t_{j-1}\}} G(t_j,t_n)^2 \int_{t_{j-1}}^{t_j} \left[e^{-a(t_j-u)}\sigma(u)\right]^2 du.
\end{aligned}$$

Expectation is given from martingale property

Recall that expectation is also known already as

$$\begin{aligned}
\mu &= \mathbb{E}^{T_1}\left[\prod_{i=1}^{k} \frac{1}{P(t_{i-1}, t_i)} \,|\, \mathcal{F}_t\right] \\
&= \frac{P(t, T_0)}{P(t, T_1)} \\
&= \prod_{i=1}^{k} \frac{P(t, t_{i-1})}{P(t, t_i)} \\
&= \prod_{i=1}^{k} \left(1 + \mathbb{E}^{t_i}\left[L_i \,|\, \mathcal{F}_t\right] \tau_i\right)
\end{aligned}$$

for $t \leq T_0$.

- Derivation can also be applied for partly fixed compounding periods with $T_0 < t \leq T_1$.

We summarise results for compounding factor terminal distribution

Lemma (OIS compounding factor distribution)

The compounding factor $\prod_{i=1}^{k}(1+L_i\tau_i) = \prod_{i=1}^{k}\frac{1}{P(t_{i-1},t_i)}$ *of an OIS coupon in Hull-White model is log-normally distributed with expectation (in T_1-forward measure)*

$$\mu = \mathbb{E}^{T_1}\left[\prod_{i=1}^{k}(1+L_i\tau_i) \mid \mathcal{F}_t\right] = \prod_{i=1}^{k}\left(1+\mathbb{E}^{t_i}\left[L_i \mid \mathcal{F}_t\right]\tau_i\right)$$

and log-normal variance

$$\nu^2 = G(T_0,T_1)^2 Var\left[x(T_0) \mid \mathcal{F}_t\right] + \sum_{j=1}^{k-1}\mathbb{1}_{\{t\le t_{j-1}\}} G(t_j,t_n)^2 \int_{t_{j-1}}^{t_j}\left[e^{-a(t_j-u)}\sigma(u)\right]^2 du.$$

Note:

- If $t \ge T_0$ then $\mathrm{Var}\left[x(T_0) \mid \mathcal{F}_t\right] = 0$.
- if $t < T_0$ then $\mathrm{Var}\left[x(T_0) \mid \mathcal{F}_t\right] = \int_t^{T_0}\left[e^{-a(T_0-u)}\sigma(u)\right]^2 du$.

Caplets and floorlets on OIS coupons can be calculated via Black formula

Theorem (OIS caplet and floorlet pricing)

A caplet or floorlet written on a compounded coupon rate $C_1 = \left\{ \left[\prod_{i=1}^{k} (1 + L_i \tau_i) \right] - 1 \right\} \frac{1}{\tau(T_0, T_1)}$ *with coupon period* $[T_0, T_1]$*, observation times* $T_0 = t_0, \ldots, t_k = T_1$ *and strike rate* K *pays at* T_1 *the payoff*

$$V(T_1) = \tau(T_0, T_1) \left[\phi \left(C_1 - K \right) \right]^+ .$$

In a Hull White model the option price at $t < T_1$ *is*

$$V(t) = P(t, T_1) \cdot Black\left(\mu, 1 + \tau(T_0, T_1) K, \nu, \phi\right)$$

with $\mu = \prod_{i=1}^{k} \left(1 + \mathbb{E}^{t_i}\left[L_i \,|\, \mathcal{F}_t\right] \tau_i\right)$ *and*

$$\begin{aligned} \nu^2 = {} & G(T_0, T_1)^2 Var\left[x(T_0) \,|\, \mathcal{F}_t\right] \\ & + \sum_{j=1}^{k-1} \mathbb{1}_{\{t \le t_{j-1}\}} G(t_j, t_n)^2 \int_{t_{j-1}}^{t_j} \left[e^{-a(t_j - u)} \sigma(u) \right]^2 du. \end{aligned}$$

Caplet and floorlet pricing formula follows directly from earlier derivations

Proof.

We abbreviate $\tau = \tau(T_0, T_1)$ and re-write the payoff as

$$V(T_1) = [\phi(\tau C_1 - \tau K)]^+ = \left[\phi\left(\left[\prod_{i=1}^{k}(1 + L_i\tau_i)\right] - (1 + \tau K)\right)\right]^+ .$$

Consequently, we can view it as an option on the compounding factor $\prod_{i=1}^{k}(1 + L_i\tau_i)$ with strike $1 + \tau(T_0, T_1)K$. Using T_1-forward measure yields the present value

$$V(t) = P(t, T_1) \cdot \mathbb{E}^{T_1}\left\{\left[\phi\left(\left[\prod_{i=1}^{k}(1 + L_i\tau_i)\right] - (1 + \tau K)\right)\right]^+ \mid \mathcal{F}_t\right\}.$$

We established earlier that the compounding factor $\prod_{i=1}^{k}(1 + L_i\tau_i)$ is log-normally distributed with expectation μ and log-normal variance ν^2 as stated in the theorem. Thus we can apply Black's formula for call and put option pricing. □

Outline

In practice, the discrete compounding factor $\prod_{i=1}^{k}(1+L_i\tau_i)$ may be approximated to simplify valuation formulas

Typically, the compounding period t_{i-1} to t_i for an overnight rate L_i is small: one day (or two/three days for holidays/weekends).
We use the short rate $r(t)$, martingale property of bank account in t_i-forward measure and approximate

$$
\begin{aligned}
1 + L_i\tau_i = \frac{1}{P(t_{i-1}, t_i)} &= \mathbb{E}^{t_i}\left[\exp\left\{\int_{t_{i-1}}^{t_i} r(u)du\right\} \mid \mathcal{F}_{t_{i-1}}\right] \\
&\approx \exp\left\{\int_{t_{i-1}}^{t_i} r(u)du\right\}.
\end{aligned}
$$

This yields continuous compounding factor approximation

$$
\prod_{i=1}^{k}(1+L_i\tau_i) \approx \prod_{i=1}^{k} e^{\int_{t_{i-1}}^{t_i} r(u)du} = e^{\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i} r(u)du} = \exp\left\{\int_{T_0}^{T_1} r(u)du\right\}.
$$

Approximate option payoff is formulated using continuous compounding factor

(Approximate) OIS caplet payoff is

$$\left[\exp\left\{\int_{T_0}^{T_1} r(u)du\right\} - [1+\tau(T_0,T_1)K]\right]^+ .$$

As before we have for $t \le T_0$

$$\begin{aligned}\mu &= \mathbb{E}^{T_1}\left[\exp\left\{\int_{T_0}^{T_1} r(u)du\right\} \mid \mathcal{F}_t\right] \\ &= \mathbb{E}^{T_1}\left[\mathbb{E}^{T_1}\left[\exp\left\{\int_{T_0}^{T_1} r(u)du\right\} \mid \mathcal{F}_{T_0}\right] \mid \mathcal{F}_t\right] \\ &= \mathbb{E}^{T_1}\left[\frac{1}{P(T_0,T_1)} \mid \mathcal{F}_t\right] = \frac{P(t,T_0)}{P(t,T_1)}.\end{aligned}$$

What is the distribution of continuous compounding factor $\exp\left\{\int_{T_0}^{T_1} r(u)du\right\}$?

We already know $I(T_0, T_1) = \int_{T_0}^{T_1} r(u)du$ from drift calculation for classical Hull White model

From the proof of Lemma lem:HW-Drift-Calibration(p. 268) we have

$$
\begin{aligned}
I(T_0, T_1) &= \int_{T_0}^{T_1} r(u)du \\
&= G(T_0, T_1) r(T_0) + \int_{T_0}^{T_1} G(u, T_1)\left[\theta(u) + \sigma(u)dW(u)\right]. \\
&= G(T_0, T_1)\left[f(0, T_0) + x(T_0)\right] + \int_{T_0}^{T_1} G(u, T_1)\left[\theta(u) + \sigma(u)dW(u)\right].
\end{aligned}
$$

This yields

- Integrated short rate $I(T_0, T_1)$ is normally distributed, thus $\exp\{I(T_0, T_1)\}$ is log-normal.
- Variance of $I(T_0, T_1)$ can be calculated via Ito isometry

$$
\bar{\nu}^2 = \mathrm{Var}\left[I(T_0, T_1) \,|\, \mathcal{F}_t\right] = G(T_0, T_1)^2 \mathrm{Var}\left[x(T_0) \,|\, \mathcal{F}_t\right] + \int_{T_0}^{T_1} \left[G(u, T)\sigma(u)\right]^2 du.
$$

With continuous rate approximation compounded rate caplet can also be priced via Black formula

Corollary

With continuous rate approximation $\prod_{i=1}^{k}(1 + L_i \tau_i) \approx \exp\left\{\int_{T_0}^{T_1} r(u)du\right\}$ *Theorem p.345 (thm:Ois-caplet-florlet-pricing) remains valid with the adjustment that log-variance* ν^2 *is replaced by* $\bar{\nu}^2$ *with*

$$\bar{\nu}^2 = G(T_0, T_1)^2 Var[x(T_0) \,|\, \mathcal{F}_t] + \int_{\max\{t, T_0\}}^{T_1} [G(u, T)\sigma(u)]^2 \, du.$$

How do log-variance ν^2 and $\bar{\nu}^2$ compare? (1/2)

We have (daily compounding)

$$\begin{aligned}
\nu^2 &= G(T_0, T_1)^2 \mathrm{Var}\left[x(T_0) \,|\, \mathcal{F}_t\right] \\
&\quad + \sum_{j=1}^{k-1} \mathbb{1}_{\{t \le t_{j-1}\}} G(t_j, t_n)^2 \int_{t_{j-1}}^{t_j} \left[e^{-a(t_j - u)}\sigma(u)\right]^2 du \\
&\approx G(T_0, T_1)^2 \mathrm{Var}\left[x(T_0) \,|\, \mathcal{F}_t\right] + \sum_{j=1}^{k-1} \mathbb{1}_{\{t \le t_{j-1}\}} G(t_j, t_n)^2 \sigma(t_j)^2 \left(t_j - t_{j-1}\right)
\end{aligned}$$

versus (continuous compounding)

$$\bar{\nu}^2 = G(T_0, T_1)^2 \mathrm{Var}\left[x(T_0) \,|\, \mathcal{F}_t\right] + \int_{\max\{t, T_0\}}^{T_1} \left[G(u, T)\sigma(u)\right]^2 du.$$

How do log-variance ν^2 and $\bar{\nu}^2$ compare? (2/2)

$$\nu^2 \approx G(T_0, T_1)^2 \mathrm{Var}\left[x(T_0) \,|\, \mathcal{F}_t\right] + \sum_{j=1}^{k-1} \mathbb{1}_{\{t \le t_{j-1}\}} G(t_j, t_n)^2 \sigma(t_j)^2 \left(t_j - t_{j-1}\right)$$

$$\bar{\nu}^2 = G(T_0, T_1)^2 \mathrm{Var}\left[x(T_0) \,|\, \mathcal{F}_t\right] + \int_{\max\{t, T_0\}}^{T_1} \left[G(u, T)\sigma(u)\right]^2 du.$$

- Variance from t to T_0, $G(T_0, T_1)^2 \mathrm{Var}\left[x(T_0) \,|\, \mathcal{F}_t\right]$, coincides in both approaches
- Variance during compounding period from T_0 to T_1 differs slightly between approaches

Log-variance ν^2 (daily compounding) can be viewed as numerical integration (or quadrature) scheme for $\bar{\nu}^2$ (continuous compounding).

Outline

Do we really need a term structure model - like Hull White model - to price caplets on compounded rates?

We establish a relation between standard (forward-looking) Libor rates and compounded (backward-looking) rates.

- Standard Libor rate with fixing time T, start time T_0 and end time T_1 (no tenor basis) is

$$L(T, T_0, T_1) = \left[\frac{P(T, T_0)}{P(T, T_1)} - 1\right] \frac{1}{\tau(T_0, T_1)}.$$

- We can define forward Libor rate $L(t, T_0, T_1)$ which *lives* for t prior to T.
- We have martingale property of forward Libor rates $L(t, T_0, T_1)$ for $t \leq T$ and well understood Vanilla models

$$dL(t,) = \sigma_L(t) \cdot dW(t)$$

(e.g. Normal model, shifted SABR model, ... - depending on choice of $\sigma_L(t)$).

How can we extend Libor rate models to compounded rates $C_1 = \left\{ \left[\prod_{i=1}^{k} (1 + L_i \tau_i)\right] - 1 \right\} \frac{1}{\tau(T_0, T_1)}$?

We generalise the definition of forward Libor rates to capture backward-looking compounded rates

Use continuous rate approximation for overnight rate, $1 + L_i \tau_i \approx \exp\left\{\int_{t_{i-1}}^{t_i} r(u)du\right\}$. This yields

$$C_1 = \left\{\exp\left\{\int_{T_0}^{T_1} r(u)du\right\} - 1\right\} \frac{1}{\tau(T_0, T_1)}$$

Define generalised forward rate

$$R(t) = \frac{1}{\tau(T_0, T_1)} \begin{cases} \left[\frac{P(t,T_0)}{P(t,T_1)} - 1\right] & t \le T_0 \\ \left[\frac{\exp\left\{\int_{T_0}^{t} r(u)du\right\}}{P(t,T_1)} - 1\right] & T_0 < t \le T_1 \end{cases}.$$

- $R(t)$ is a martingale in T_1-forward measure (by construction).
- $R(t)$ coincides with standard forward Libor rate $L(t, T_0, T_1)$ for all t until fixing time T.
- $R(T_1)$ is equal to compounded rate C_1.

Now we can specify a Vanilla model for the generalised forward rate

We specify a Vanilla model for the generalised forward rate as

$$dR(t) = \sigma_R(t) \cdot dW(t).$$

Here, $W(t)$ is a Brownian motion in T_1-forward measure and $\sigma_R(t)$ is an adapted volatility process.

How can we specify volatility $\sigma_R(t)$?

For $t \leq T$ $R(t) = L(t, T_0, T_1)$, thus also $dR(t) = dL(t,)$.

- We use standard Libor rate volatility $\sigma_R(t) = \sigma_L(t)$ for $t \leq T$.
- But what can we do for $T_0 < t \leq T_1$?

We need to take into account that between T_0 and T_1 more and more overnight rates get fixed

- At observation time $t \to T_1$ we get that $r(u)$, with $u \leq t$ in $C_1 = \left\{ \exp\left\{ \int_{T_0}^{T_1} r(u)du \right\} - 1 \right\} \frac{1}{\tau(T_0,T_1)}$ is deterministic.
- Volatility of coupon decreases to zero as $t \to T_1$.

Assume linear decay of volatility of generalised forward rates,

$$\sigma_R(t) = \frac{T_1 - t}{T_1 - T_0} \cdot \sigma(t), \qquad T_0 < t \leq T_1.$$

For backbone volatility $\sigma(t)$ we can use same type of model as for Libor volatility $\sigma_L(t)$.

Let's have a look at a simple example Vanilla model with normal dynamics and constant volatility

$$dR(t) = \min\left\{1, \frac{T_1 - t}{T_1 - T_0}\right\} \cdot \sigma \cdot dW(t).$$

- Final rate $R(T_1) = C_1$ is normally distributed. Option on C_1 can be priced with Bachelier formula
- Integrated variance of C_1 at observation (pricing) time $t < T_0$ becomes

$$\begin{aligned}\nu^2 &= \int_t^{T_1} \left[\min\left\{1, \frac{T_1 - t}{T_1 - T_0}\right\} \cdot \sigma\right]^2 dt \\ &= \sigma^2 \cdot (T_0 - t) + \frac{1}{3}\sigma^2 (T_1 - T_0) \cdot\end{aligned}$$

- Analogous derivation holds for shifted Log-normal model for $R(t)$
- Compare with integrated variance in Hull-White model for mean reversion $a \to 0$!

Outline

We can re-use Vanilla and term structure models to price caps and floors on compounded rate coupons

- Compounded overnight rate coupon rates are

$$C_1 = \left\{ \left[\prod_{i=1}^{k} (1 + L_i \tau_i) \right] - 1 \right\} \frac{1}{\tau} \approx \left\{ \exp\left\{ \int_{T_0}^{T_1} r(u)du \right\} - 1 \right\} \frac{1}{\tau}$$

- Terminal distribution of C_1 and caplets/floorlets on C_1 can be calculated using Hull-White model
- A generalisation of Libor forward rates to the compounding period T_0 to T_1 yields generalised forward rates $R(t)$ for which we can specify Vanilla models

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