Interest Rate Modelling and Derivative Pricing

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Part I

Introduction and Preliminaries

Outline

Outline

Introduction and Agenda

Stochastic Calculus Basics

Basic Fixed Income Modelling

What is this lecture about?

Interbank swap deal example

Pays 3% on 100mm EUR

Start date: Oct 30, 2020

End date: Oct 30, 2040

(annually, 30/360 day count, modified following, Target calendar)

Pays 6-months Euribor floating rate on 100mm EUR

Start date: Oct 30, 2020

End date: Oct 30, 2040

(semi-annually, act/360 day count, modified following, Target calendar)

Suppose, Bank A may decide to early terminate deal in 10, 11, 12,.. years

How does early termination option affect the present value and risk of the deal?

Organisational details first

- Lecture: Fri, 9:15 - 10:45 s.t., RUD25, R 1.012 (plus some additional times)
- Exercises: Fri, 11:00 - 12:30, RUD25, R 1.012 (every second week, some exceptions)
- Office times: Fridays on request before or after the lecture

Exercises:

- Discuss and analyse practical examples and theory details
- Main tool: QuantLib (open source financial library)
- Implementation: Python, some Excel

Requirements:

- Present at least once during exercises
- exam planned for July 25, 2025

Literature and resources you will need

- Literature
  - L. Andersen and V. Piterbarg. *Interest rate modelling, volume I to III.*
    Atlantic Financial Press, 2010
  - D. Brigo and F. Mercurio. *Interest Rate Models - Theory and Practice.*
    Springer-Verlag, 2007
  - S. Shreve. *Stochastic Calculus for Finance II - Continuous-Time Models.*
    Springer-Verlag, 2004
- QuantLib web site `www.quantlib.org`
- Official source repository `www.github.com/lballabio`
- Some extensions which we might use
  `www.github.com/sschlenkrich`
- `https://www.applied-financial-mathematics.de/`
  `interest-rate-modelling-and-derivative-pricing-summer-term-2`

Let's revisit the introductory example

Interbank swap deal example

Fixed interest rate

Pays 3% on 100mm EUR

Notional

Start date: Oct 30, 2020

Dates

End date: Oct 30, 2040

Market conventions

(annually, 30/360 day count, modified following, Target calendar)

Stochastic interest rates

Pays 6-months Euribor floating rate on 100mm EUR

Start date: Oct 30, 2020

End date: Oct 30, 2040

(semi-annually, act/360 day count, modified following, Target calendar)

Optionalities

Bank A may decide to early terminate deal in 10, 11, 12,.. years

Agenda covers static yield curve modelling, Vanilla rates models and term structure models

Interest Rate Modelling

- Stochastic calculus basics
- Static yield curve modelling and linear products
- Vanilla interest rate models
- HJM term structure modelling framework
- Classical Hull-White interest rate model
- Pricing methods for Bermudan swaptions

Model Calibration

- Multi-curve yield curve calibration
- Hull-White model calibration
- Numerical methods for model calibration

Sensitivity Calculation

- Delta and Vega specification
- Numerical methods for sensitivity calculation

Outline

We will work along three streams

Probability space & filtration	Brownian Motion	Self-financing trading strategy & arbitrage
Change of measure	Ito integral	Equivalent martingale measure & FTAP
Martingale	Martingale representation	Change of equiv. martingale meas.
Density process	Ito's lemma	Permissible trading strategy

Risk-neutral derivative pricing formula

Outline

Measure theory is independent of financial application

Probability space & filtration	Brownian Motion	Self-financing trading strategy & arbitrage
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Risk-neutral derivative pricing formula

We start with stochastic processes and probability space

Stochastic process (for assets or interest rate components)

$$X(t) = [X_1(t), \ldots, X_p(t)]^\top .$$

Probability space that drives stochastic process $(\Omega, \mathcal{F}, \mathbb{P})$

- Ω sample space with outcomes ω (typically increments of Brownian motions),
- $\mathcal{F}$ σ-algebra on Ω,
- $\mathbb{P}$ probability measure on $\mathcal{F}$.

Information flow is realised via filtration $\{\mathcal{F}_t,\ t \in [0, T]\}$

- $\mathcal{F}_t$ sub-algebra of $\mathcal{F}$ with $\mathcal{F}_t \subseteq \mathcal{F}_s$ for $t \leq s$,
- Assume $X(t)$ is adapted to filtration $\mathcal{F}_t$, i.e. $X(t)$ is fully observable at time t.

Measures can be linked by Radon–Nikodym derivative

Theorem (Radon–Nikodym derivative)

Let $\mathbb{P}$ and $\hat{\mathbb{P}}$ be equivalent probability measures on $(\Omega, \mathcal{F})$. Then there exists a unique (a.s.) non-negative random variable $R(\omega)$ with $\mathbb{E}^{\mathbb{P}}[R] = 1$, such that for all $A \in \mathcal{F}$

$$\hat{\mathbb{P}}(A) = \mathbb{E}^{\mathbb{P}}\left[R\, \mathbb{1}_{\{A\}}\right].$$

R is denoted Radon–Nikodym derivative.

It follows

$$\hat{\mathbb{P}}(A) = \int_A d\hat{\mathbb{P}} = \int_A R\, d\mathbb{P} = \mathbb{E}^{\mathbb{P}}\left[R\, \mathbb{1}_{\{A\}}\right].$$

and also for all measurable functions X (via algebraic induction)

$$\mathbb{E}^{\hat{\mathbb{P}}}[X] = \mathbb{E}^{\mathbb{P}}[R\, X].$$

Thus we may write

$$R = d\hat{\mathbb{P}}/d\mathbb{P}.$$

We will frequently need the change of measure for conditional expectations

Definition (Conditional expectation)

Let X be a random variable. The conditional expectation $\mathbb{E}^{\mathbb{P}}[X \mid \mathcal{F}_t]$ is defined as the stochastic variable that satisfies:

- $\mathbb{E}^{\mathbb{P}}[X \mid \mathcal{F}_t]$ is $\mathcal{F}_t$-measurable and
- for all $A \in \mathcal{F}_t$ we have

$$\int_A \mathbb{E}^{\mathbb{P}}[X \mid \mathcal{F}_t] \, d\mathbb{P} = \int_A X d\mathbb{P}.$$

Theorem (Baye's rule for conditional expectation)

Let $R = d\hat{\mathbb{P}}/d\mathbb{P}$ be the Radon–Nikodym derivative associated with $(\Omega, \mathcal{F}, \mathbb{P})$ and $\left(\Omega, \mathcal{F}, \hat{\mathbb{P}}\right)$ and X a random variable. Then

$$\mathbb{E}^{\hat{\mathbb{P}}}[X \mid \mathcal{F}_t] = \frac{\mathbb{E}^{\mathbb{P}}[R\,X \mid \mathcal{F}_t]}{\mathbb{E}^{\mathbb{P}}[R \mid \mathcal{F}_t]}.$$

We sketch the proof for change of measure (1/2)

We use the definition of conditional expectation and show that (for all $A \in \mathcal{F}_t$)

$$\int_A \mathbb{E}^{\mathbb{P}}[R\,X \mid \mathcal{F}_t]\, d\mathbb{P} = \int_A \mathbb{E}^{\mathbb{P}}[R \mid \mathcal{F}_t]\, \mathbb{E}^{\hat{\mathbb{P}}}[X \mid \mathcal{F}_t]\, d\mathbb{P}.$$

We have for the left side using conditional expectation and Radon–Nikodym derivative

$$\int_A \mathbb{E}^{\mathbb{P}}[R\,X \mid \mathcal{F}_t]\, d\mathbb{P} = \int_A X\,R\, d\mathbb{P} = \int_A X d\hat{\mathbb{P}}.$$

For the right side we get using conditional expectation

$$\begin{aligned}\int_A \mathbb{E}^{\mathbb{P}}[R \mid \mathcal{F}_t]\, \mathbb{E}^{\hat{\mathbb{P}}}[X \mid \mathcal{F}_t]\, d\mathbb{P} &= \int_A \mathbb{E}^{\mathbb{P}}\left[\mathbb{E}^{\hat{\mathbb{P}}}[X \mid \mathcal{F}_t]\, R \mid \mathcal{F}_t\right] d\mathbb{P} \\ &= \int_A \mathbb{E}^{\hat{\mathbb{P}}}[X \mid \mathcal{F}_t]\, R\, d\mathbb{P}.\end{aligned}$$

We sketch the proof for change of measure (2/2)

Applying Radon–Nikodym derivative and again conditional expectation yields

$$\int_A \mathbb{E}^{\hat{\mathbb{P}}}\left[X \mid \mathcal{F}_t\right] R \, d\mathbb{P} = \int_A \mathbb{E}^{\hat{\mathbb{P}}}\left[X \mid \mathcal{F}_t\right] d\hat{\mathbb{P}} = \int_A X d\hat{\mathbb{P}}.$$

We will use Frobenius norm in martingale definition

Sum of squares notation (Frobenius norm, L^2 norm for vectors)

For a matrix or vector $A \in \mathbb{R}^{m \times n}$ with elements $\{a_{i,j}\}_{i,j}$ we denote

$$|A| = \sqrt{\operatorname{tr}\left(AA^\top\right)} = \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} a_{i,j}^2}.$$

Martingales allow derivation of expected future values

Definition (Martingale)

Let $X(t)$ be an adapted vector-valued process with finite absolute expectation $\mathbb{E}^{\mathbb{P}}[|X(t)|] < \infty$ (under the probability measure $\mathbb{P}$) for all $t \in [0, T]$.
$X(t)$ is a martingale under $\mathbb{P}$ if for all $t, s \in [0, T]$ with $t \leq s$

$$X(t) = \mathbb{E}^{\mathbb{P}}[X(s) \mid \mathcal{F}_t] \quad a.s.$$

- Typically, martingale property is derived (by other results) for a process.
- Then we can use martingale property to obtain expectation of future values $X(T)$.

Density process describes change of measure for processes

Definition (Density process)

Denote $\zeta(t) = \mathbb{E}^{\mathbb{P}}\left[d\hat{\mathbb{P}}/d\mathbb{P} \mid \mathcal{F}_t\right]$ the density process of $\hat{\mathbb{P}}$ (relative to $\mathbb{P}$).

- Then $\zeta(t)$ is a $\mathbb{P}$-martingale with $\zeta(0) = \mathbb{E}^{\mathbb{P}}[\zeta(t)] = 1$.

Lemma (Change of measure for processes)

Let $X(t)$ be a $\mathcal{F}_t$ measurable random variable. Then

$$\mathbb{E}^{\hat{\mathbb{P}}}[X(T) \mid \mathcal{F}_t] = \mathbb{E}^{\mathbb{P}}\left[\frac{\zeta(T)}{\zeta(t)}X(T) \mid \mathcal{F}_t\right].$$

Proof.

Recall that $R = d\hat{\mathbb{P}}/d\mathbb{P}$. We have $\mathbb{E}^{\hat{\mathbb{P}}}[X(T) \mid \mathcal{F}_t] = \frac{\mathbb{E}^{\mathbb{P}}[R\,X(T) \mid \mathcal{F}_t]}{\mathbb{E}^{\mathbb{P}}[R \mid \mathcal{F}_t]}$. Then

$$\mathbb{E}^{\mathbb{P}}[R\,X(T) \mid \mathcal{F}_t] = \mathbb{E}^{\mathbb{P}}\left[\mathbb{E}^{\mathbb{P}}[R\,X(T) \mid \mathcal{F}_T] \mid \mathcal{F}_t\right] = \mathbb{E}^{\mathbb{P}}\left[\mathbb{E}^{\mathbb{P}}[R \mid \mathcal{F}_T]\,X(T) \mid \mathcal{F}_t\right].$$

The result follows from the definition of $\zeta(t)$ via $\zeta(t) = \mathbb{E}^{\mathbb{P}}[R \mid \mathcal{F}_t]$. □

Density process may be used to define a new measure

Let $\zeta(t)$ be a $\mathbb{P}$-martingale with $\zeta(0) = 1$. We choose a final horizon time T and define the Radon–Nikodym derivative as $R(\omega) = \zeta(T, \omega)$.

The corresponding measure $\hat{\mathbb{P}}$ on $(\Omega, \mathcal{F}_T)$ is

$$\hat{\mathbb{P}}(A) = \mathbb{E}^{\mathbb{P}}\left[R\,1_{\{A\}}\right] = \mathbb{E}^{\mathbb{P}}\left[\zeta(T,\omega)\,\mathbb{1}_{\{A\}}\right].$$

We show that the density of $\hat{\mathbb{P}}$ indeed equals $\zeta(t)$.

Denote $\bar{\zeta}(t) = \mathbb{E}^{\mathbb{P}}\left[R \,|\, \mathcal{F}_t\right]$ the density of $\hat{\mathbb{P}}$. Then we get with the martingale property of $\zeta(t)$

$$\bar{\zeta}(t) = \mathbb{E}^{\mathbb{P}}\left[\zeta(T,\omega) \,|\, \mathcal{F}_t\right] = \zeta(t).$$

Outline

Diffusion processes are the basis for our models

Probability space & filtration	Brownian Motion	Self-financing trading strategy & arbitrage
Change of measure	Ito integral	Equivalent martingale measure & FTAP
Martingale	Martingale representation	Change of equiv. martingale meas.
Density process	Ito's lemma	Permissible trading strategy

Risk-neutral derivative pricing formula

Stochastic process is driven by Brownian motion

Information is generated by Brownian motion

- $W(t) = [W_1(t), \ldots, W_d(t)]^\top$ d-dimensional Brownian motion.
- $W_i(\cdot)$ independent of $W_j(\cdot)$ for $i \neq j$.
- Independent Gaussian increments $W_i(s) - W_i(t) \sim \mathcal{N}(0, s-t)$ for $s \geq t$.
- Typically, filtration $\mathcal{F}_t$ is generated by Brownian motion $W(\cdot)$, i.e. $\mathcal{F}_t = \sigma\{W(u), 0 \leq u \leq t\}$.

Definition (H^2 for volatility processes σ)

Let $\sigma : \mathbb{R} \times \Omega \to \mathbb{R}^{p \times d}$ be a volatility process adapted to the filtration generated by $\mathcal{F}_t$. We say that σ is in H^2 if for all $t \in [0, T]$ we have

$$\mathbb{E}^{\mathbb{P}}\left[\int_0^t |\sigma(s,\omega)|^2 \, ds\right] < \infty.$$

Stochastic process is described as Ito process with Ito integral

$$X(t) = X(0) + \int_0^t \mu(s,\omega)\, ds + \int_0^t \sigma(s,\omega)\, dW(s)$$

or in differential notation

$$dX(t) = \mu(t,\omega)\, dt + \sigma(t,\omega)\, dW(t),$$

- vector-valued drift $\mu : \mathbb{R} \times \Omega \to \mathbb{R}^p$,
- matrix of volatilities $\sigma : \mathbb{R} \times \Omega \to \mathbb{R}^{p\times d}$,
- assume drift μ and volatility σ are adapted to $\mathcal{F}_t$ and σ is in H^2.

We consider the Ito integral as

$$\int_0^t \sigma(s,\omega)\, dW(s) = \lim_{n\to\infty} \sum_{i=1}^{n} \sigma(s_{i-1},\omega)\left[W(s_i) - W(s_{i-1})\right], \quad s_i = \frac{i}{n}t.$$

Ito integrals are important martingales for modelling

Theorem (Ito Integral properties)

Define the Ito integral $X(t) = \int_0^t \sigma(u,\omega)\, dW(u)$ *with* σ *is in* H^2. *Then*

1. $X(t)$ *is* $\mathcal{F}_t$*-measurable (i.e. we can calculate the distribution of* $X(t)$ *using* $(\Omega, \mathcal{F}, \mathbb{P})$*)*

2. $X(t)$ *is a continuous martingale*

3. $\mathbb{E}^{\mathbb{P}}\left[|X(t)|^2\right] = \mathbb{E}^{\mathbb{P}}\left[\int_0^t |\sigma(u,\omega)|^2\, du\right] < \infty$ *(Ito isometry)*

4. $\mathbb{E}^{\mathbb{P}}\left[X(t)X(s)^\top\right] = \mathbb{E}^{\mathbb{P}}\left[\int_0^{\min\{t,s\}} \sigma(u,\omega)\,\sigma(u,\omega)^\top\, dt\right]$ *(auto-covariance)*

Stochastic processes can be represented as Ito integrals

Theorem (Martingale representation theorem)

If $X(\cdot)$ is a (local) martingale adapted to the filtration $\mathcal{F}_t$ which is generated by Brownian motion $W(\cdot)$ then there exists a volatility process $\sigma(t,\omega)$ such that

$$dX(t) = \sigma(t,\omega)\, dW(t).$$

Moreover, if $X(\cdot)$ is a square-integrable martingale then σ is in H^2.

Ito's Lemma is one of the most relevant tools

Theorem (Ito's Lemma)

Let $X(t)$ be an Ito process and $f(\cdot)$ a twice continuous differentiable scalar function. Then

$$df\left(X(t)\right) = \nabla_X f(X)^\top dX(t) + \frac{1}{2} dX(t)^\top H_X f(x) dX(t)$$

with $\nabla_X f$ being the gradient of f and $H_X f(x)$ being the Hessian of f.

Here we use *calculus* $dW_i(t)dW_i(t) = dt$ and $dW_i(t)dW_j(t) = 0$ for $i \neq j$.

Corollary (Ito product rule)

Let $X_1(t)$ and $X_2(t)$ be scalar Ito processes. Then

$$d\left[X_1(t)X_2(t)\right] = X_1(t)dX_2(t) + X_2(t)dX_1(t) + dX_1(t)dX_2(t).$$

Outline

Pricing builds on measure theory and stochastic processes

Probability space & filtration	Brownian Motion	Self-financing trading strategy & arbitrage
Change of measure	Ito integral	Equivalent martingale measure & FTAP
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Density process	Ito's lemma	Permissible trading strategy

Risk-neutral derivative pricing formula

We specify our market based on assets and trading strategies

Financial Market

We assume p (dividend-free[1]) assets $X(t) = [X_1(t), \ldots, X_p(t)]^\top$ which are driven by Ito processes

$$dX(t) = \mu(t, \omega)\, dt + \sigma(t, \omega)\, dW(t).$$

Trading Strategy

A trading strategy represents a predictable adapted process (of asset weights)

$$\phi(t, \omega) = [\phi_1(t, \omega), \ldots, \phi_p(t, \omega)]^\top .$$

The value of the trading strategy (or corresponding portfolio) is

$$\pi(t) = \phi(t)^\top X(t).$$

[1] I.e. no intermediate payments

Self-financing strategies and arbitrage

Trading Gains and Self-financing Strategy

Trading gains (over a short period of time) are $\phi(t)^\top [X(t+dt) - X(t)]$.
This leads to the general specification $\int_t^T \phi(t)^\top dX(t)$.
A trading strategy is self-financing if portfolio changes are only induced by asset returns (no money inflow or outflow). That is

$$\pi(T) - \pi(t) = \int_t^T \phi(s)^\top dX(s).$$

Definition (Arbitrage)

An arbitrage opportunity is a self-financing strategy $\phi(\cdot)$ with $\pi(0) = 0$ and, for some $t \in [0, T]$,

$$\pi(t) \geq 0 \text{ a.s., and } \mathbb{P}(\pi(t) > 0) > 0.$$

Arbitrage needs to be precluded in a financial model.

Absence of arbitrage is closely related to equivalent martingale measures

Definition (Numeraire and equivalent martingale measure)

A numeraire is a positive asset $N(t)$ of our market. An equivalent martingale measure (corresponding to the numeraire $N(t)$) is a measure $\mathbb{Q}$ such that the normalised asset prices $[X_1(t)/N(t), \ldots, X_p(t)/N(t)]^\top$ are $\mathbb{Q}$-martingales.

Fundamental theorem of asset pricing

Assuming some restrictions on permissible trading strategies one can show that absence of arbitrage is "nearly equivalent" to the existence of an equivalent martingale measure.

Our models are all based on the assumption of no-arbitrage and the existence of an equivalent martingale measure.

Equivalent martingale measures exists for any numeraire (1/2)

Suppose we have a numeraire $N(t)$ and an equivalent martingale measure $\mathbb{Q}^N$. Suppose we also have another numeraire $M(t)$. Define

$$\zeta(t) = \frac{M(t)}{N(t)} \frac{N(0)}{M(0)}.$$

Then

- $\mathbb{E}^N \left[\zeta(T) \,|\, \mathcal{F}_t \right] = \mathbb{E}^N \left[\frac{M(T)}{N(T)} \,|\, \mathcal{F}_t \right] \frac{N(0)}{M(0)} = \frac{M(t)}{N(t)} \frac{N(0)}{M(0)} = \zeta(t)$, thus $\zeta(t)$ is a $\mathbb{Q}^N$-martingale
- $\zeta(0) = \frac{M(0)}{N(0)} \frac{N(0)}{M(0)} = 1$

Equivalent martingale measures exists for any numeraire (2/2)

Define the new measure $\mathbb{Q}^M$ via the density $\zeta(t)$. Then for an asset $X_i(t)$

$$\mathbb{E}^M\left[\frac{X_i(T)}{M(T)}\,|\,\mathcal{F}_t\right] = \mathbb{E}^N\left[\frac{\zeta(T)}{\zeta(t)}\,\frac{X_i(T)}{M(T)}\,|\,\mathcal{F}_t\right] = \mathbb{E}^N\left[\frac{M(T)}{N(T)}\,\frac{N(\mathrm{t})}{M(t)}\,\frac{X_i(T)}{M(T)}\,|\,\mathcal{F}_t\right].$$

Taking out what is known and using the martingale property of measure $\mathbb{Q}^N$ yields

$$\mathbb{E}^M\left[\frac{X_i(T)}{M(T)}\,|\,\mathcal{F}_t\right] = \frac{N(\mathrm{t})}{M(t)}\,\mathbb{E}^N\left[\frac{X_i(T)}{N(T)}\,|\,\mathcal{F}_t\right] = \frac{N(\mathrm{t})}{M(t)}\,\frac{X_i(t)}{N(t)} = \frac{X_i(t)}{M(t)}.$$

$X_i(t)/M(t)$ is a $\mathbb{Q}^M$-martingale. Thus $\mathbb{Q}^M$ is an equivalent martingale measure for $M(t)$.

Trading strategies need to be permissible

Definition (Permissible trading strategy)

Let $X(t)$ be an Ito process and $\mathbb{Q}$ an equivalent martingale measure with numeraire $N(t)$. A self-financing trading strategy $\phi(t)$ is called permissible if

$$\int_0^t \phi(s)^\top d\left(\frac{X(s)}{N(s)}\right)$$

is a $\mathbb{Q}$-martingale.

Recall that $X(t)/N(t)$ is a $\mathbb{Q}$-martingale by construction. If $\phi(t)$ is sufficiently bounded then it is also permissible.

Theorem (Martingale property for trading strategies)

For any self-financing and permissible trading strategy $\phi(t)$ and an equivalent martingale measure $\mathbb{Q}$ with numeraire $N(t)$ the discounted portfolio price process $\pi(t)/N(t)$ is a martingale.

On average you can not beat the market when trading in the assets.

We proof the martingale property for trading strategies

Proof.

Recall that $\pi(t) = \phi(t)^\top X(t)$. The self-financing condition may be written as $d\pi(t) = \phi(t)^\top dX(t)$. Applying Ito's product rule yields

$$
\begin{aligned}
d\left[\frac{\pi(t)}{N(t)}\right] &= d\left[\pi(t)\frac{1}{N(t)}\right] = \frac{d\pi(t)}{N(t)} + \pi(t)d\left[\frac{1}{N(t)}\right] + d\pi(t)d\left[\frac{1}{N(t)}\right] \\
&= \frac{\phi(t)^\top dX(t)}{N(t)} + \phi(t)^\top X(t)d\left[\frac{1}{N(t)}\right] + \phi(t)^\top dX(t)d\left[\frac{1}{N(t)}\right] \\
&= \phi(t)^\top \left[\frac{dX(t)}{N(t)} + X(t)d\left[\frac{1}{N(t)}\right] + dX(t)d\left[\frac{1}{N(t)}\right]\right] \\
&= \phi(t)^\top d\left[\frac{X(t)}{N(t)}\right].
\end{aligned}
$$

Now the assertion follows directly from the condition that $\phi(t)$ is permissible. □

Derivative pricing is closely related to trading strategies

Definition (Contingent claim)

A derivative security (or contingent claim) pays at time T the random variable $V(T)$ (no intermediate payments). We assume $V(T)$ has finite variance and is attainable. That is there exists a permissible trading strategy $\phi(\cdot)$ such that

$$V(T) = \phi(T)^\top X(T) \, a.s.$$

Then absence of arbitrage yields that the fair price $V(t)$ of the derivative security becomes

$$V(t) = \phi(t)^\top X(t) \text{ for all } t \in [0, T].$$

Consequently,

$$\frac{V(t)}{N(t)} = \frac{\phi(t)^\top X(t)}{N(t)} = \mathbb{E}^{\mathbb{Q}}\left[\frac{\phi(T)^\top X(T)}{N(T)} \,|\, \mathcal{F}_t\right] = \mathbb{E}^{\mathbb{Q}}\left[\frac{V(T)}{N(T)} \,|\, \mathcal{F}_t\right].$$

Above arbitrage pricing formula is the foundation of derivative pricing.

Outline

We summarize the key results

Probability space & filtration	Brownian Motion	Self-financing trading strategy & arbitrage
Change of measure	Ito integral	Equivalent martingale measure & FTAP
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Risk-neutral derivative pricing formula

We summarize the key results (cheat sheet)

$$(\Omega, \mathcal{F}, \mathbb{P}),\ \mathcal{F}_t,\ t \in [0, T]$$

$$W(t) = [W_1(t), \ldots, W_d(t)]^\top$$

$$d\pi(T) = \phi(t)^\top dX(t)$$

$$\mathbb{E}^{\hat{\mathbb{P}}}[X \mid \mathcal{F}_t] = \frac{\mathbb{E}^{\mathbb{P}}[R\,X \mid \mathcal{F}_t]}{\mathbb{E}^{\mathbb{P}}[R \mid \mathcal{F}_t]}$$

$$X(t) = \int_0^t \sigma(u, \omega)\, dW(u)$$

$$\frac{X(t)}{N(t)} = \mathbb{E}^{\mathbb{Q}}\left[\frac{X(T)}{N(T)} \mid \mathcal{F}_t\right]$$

$$X(t) = \mathbb{E}^{\mathbb{P}}[X(s) \mid \mathcal{F}_t]$$

$$dX(t) = \sigma(u, \omega)\, dW(u)$$

$$\mathbb{E}^M\left[\frac{X_i(T)}{M(T)} \mid \mathcal{F}_t\right] = \mathbb{E}^N\left[\frac{N(t)}{M(t)}\, \frac{X_i(T)}{N(T)} \mid \mathcal{F}_t\right]$$

$$\zeta(t) = \mathbb{E}^{\mathbb{P}}\left[d\hat{\mathbb{P}}/d\mathbb{P} \mid \mathcal{F}_t\right]$$

$$df = f'\,dX + \frac{f''}{2}\,dX^2$$

$$\phi(t)^\top d\left[\frac{X(t)}{N(t)}\right] = \bar{\sigma}\, dW(t)$$

$$V(t)/N(t) = \mathbb{E}^{\mathbb{Q}}[V(T)/N(T) \mid \mathcal{F}_t]$$

Outline

Outline

First we need to specify the assets in the market (1/2)

Example (Overnight bank account)

- Suppose bank A deposits 1 EUR at ECB at time $T_0 = 0$ (today) with the right to withdraw money at T_1, say the next day.
- Bank A may leave deposit with ECB as long as they want
- Time T_i is measured in years (or year fraction) for simplicity
- ECB pays annualized interest rate r_i from T_i to T_{i+1}

Example also holds for deposits between two banks, e.g. bank A and bank B.

What is the value of the deposit at a future time T_N?

First we need to specify the assets in the market (2/2)

Denote B_i the value of the deposit at time T_i. Then

$$B_0 = 1$$

and

$$B_i = B_{i-1} + r_{i-1} \cdot (T_i - T_{i-1}) \cdot B_{i-1} = [1 + r_{i-1}(T_i - T_{i-1})] \cdot B_{i-1}.$$

The most basic asset is the money market bank account

Definition (Short rate and (abstract) bank account)

Assume a process $r(t)$ (adapted to the filtration $\mathcal{F}_t$) for the instantaneous interest rate. The rate $r(t)$ is denoted the short rate.
The continuous compounded bank account (or money market account) is an asset with price $B(t)$ given by $B(0) = 1$ and

$$dB(t) = r(t) \cdot B(t) \cdot dt.$$

It follows that the future price of the bank account becomes

$$B(t) = \exp\left\{ \int_0^t r(s)ds \right\}.$$

Short rate $r(t)$ is considered the *risk-free rate* at which market participants can lend and borrow money.

The most relevant assets are zero coupon bonds (ZCBs) (1/2)

ZCBs are fixed future cash flows of unit notional, e.g. 1 EUR in 10y.

Definition (Zero Coupon Bond)

A zero coupon bond for maturity T is an asset with time-t asset price $P(t, T)$ for $t \leq T$ and $P(T, T) = 1$.

What is the time-t asset price of a zero coupon bond?

The most relevant assets are zero coupon bonds (ZCBs) (2/2)

Use risk-neutral pricing formula!

Select money market account $B(t)$ as numeraire and denote $\mathbb{Q}$ the equivalent martingale measure.

Then (with $\mathbb{E}_t^{\mathbb{Q}}[\cdot] = \mathbb{E}^{\mathbb{Q}}[\cdot|\mathcal{F}_t]$)

$$\frac{P(t,T)}{B(t)} = \mathbb{E}_t^{\mathbb{Q}}\left[\frac{P(T,T)}{B(T)}\right] = \mathbb{E}_t^{\mathbb{Q}}\left[B(T)^{-1}\right] = \mathbb{E}_t^{\mathbb{Q}}\left[\exp\left\{-\int_0^T r(s)ds\right\}\right].$$

Multiplying with $B(t) = \exp\left\{\int_0^t r(s)ds\right\}$ yields

$$P(t,T) = \mathbb{E}_t^{\mathbb{Q}}\left[\exp\left\{-\int_t^T r(s)ds\right\}\right].$$

And what is the ZCB price in terms of money ...?

- Formula $P(t,T) = \mathbb{E}_t^{\mathbb{Q}}\left[\exp\left\{-\int_t^T r(s)ds\right\}\right]$ is a model-independent result
- To calculate it more concrete we need to specify a model/dynamics for short rate $r(t)$
- Suppose short rate is known deterministic function, then

$$P(t,T) = \exp\left\{-\int_t^T r(s)ds\right\}.$$

- Suppose short rate is fixed, i.e. $r(t) = r_0$, then (even simpler)

$$P(t,T) = e^{-r_0(T-t)}.$$

For our market we assume that today's prices $P(0,T)$ of all ZCBs (with maturity $T \geq 0$) are known.

Interest rate market consists of money market bank account and zero coupon bonds

Interest rate market

We consider a market consisting of the money market account $B(t)$ and zero coupon bonds $P(t, T)$ for $t \leq T$ as financial assets.

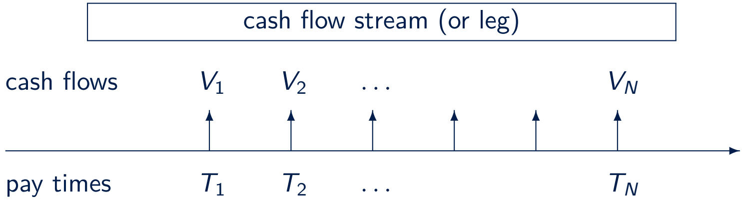
Interest rate derivatives

Interest rate derivatives are contingent claims (or baskets of contingent claims) depending on realisations of future zero coupon bonds.

- We may restrict modelling to discrete set of ZCBs $\{P(t, T_i)\}_i$ (vanilla models).
- Full continuum of ZCBs $\{P(t, T) \,|\, t \leq T\}$ is modelled via term structure models.

Outline

Discounted cash flow (DCF) pricing methodology ...



$$\frac{V(t)}{B(t)} = \sum_{i=1}^{N} \mathbb{E}^{\mathbb{Q}} \left[\frac{V_i}{B(T_i)} \,|\, \mathcal{F}_t \right]$$

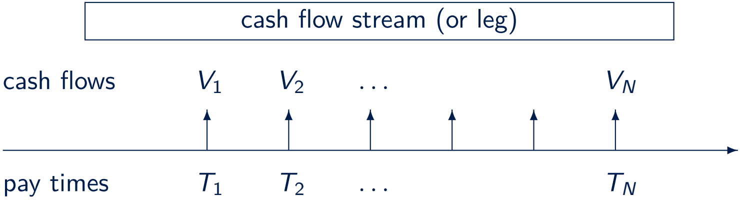
Denote $\mathbb{E}^{T_i}[\cdot]$ expectations in T_i-forward measures with zero coupon bond numeraire $P(t, T_i)$ $(i = 1, \ldots, N)$. Then (change of measure)

$$\frac{V(t)}{B(t)} = \sum_{i=1}^{N} \mathbb{E}^{T_i} \left[\frac{P(t, T_i)}{B(t)} \cdot \frac{V_i}{P(T_i, T_i)} \,|\, \mathcal{F}_t \right].$$

With $P(T_i, T_i) = 1$ follows

$$V(t) = \sum_{i=1}^{N} P(t, T_i) \cdot \mathbb{E}^{T_i} \left[V_i \,|\, \mathcal{F}_t \right].$$

(DCF) ... is a model-independent concept



$$V(t) = \sum_{i=1}^{N} P(t, T_i) \cdot \mathbb{E}^{T_i} \left[V_i \,|\, \mathcal{F}_t \right]$$

- Present value is sum of discounted expected future cash flows.
- If future cash flows are known (i.e. deterministic), then

$$V(t) = \sum_{i=1}^{N} P(t, T_i) \cdot V_i$$

.

- In general, challenge lies in calculating $\mathbb{E}^{T_i} \left[V_i \,|\, \mathcal{F}_t \right]$ using a model.

Part II

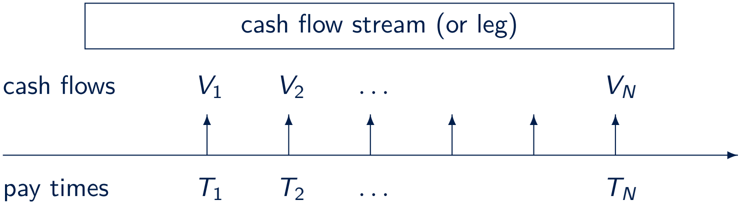
Yield Curves and Linear Products

Outline

Outline

Outline

DCF method requires knowledge of today's ZCB prices



- Assume $t = 0$ and deterministic cash flows, then

$$V(0) = \sum_{i=1}^{N} P(0, T_i) \cdot V_i .$$

How do we get today's ZCB prices $P(0, T_i)$?

Yield curve is fundamental object for interest rate modelling

- A yield curve (YC) at an observation time t is the function of zero coupon bonds $P(t, \cdot) : [t, \infty) \to \mathbb{R}^+$ for maturities $T \geq t$.

- YCs are typically represented in terms of interest rates (instead of zero coupon bond prices).

- Discretely compounded zero rate curve $z_p(t, T)$ with frequency p, such that
$$P(t, T) = \left(1 + \frac{z_p(t, T)}{p}\right)^{-p \cdot (T-t)}.$$

- Simple compounded zero rate curve $z_0(t, T)$ (i.e. $p = 1/(T-t)$), such that
$$P(t, T) = \frac{1}{1 + z_0(t, T) \cdot (T - t)}.$$

- Continuous compounded zero rate curve $z(t, T)$ (i.e. $p = \infty$), such that
$$P(t, T) = \exp\left\{-z(t, T) \cdot (T - t)\right\}.$$

For interest rate modelling we also need continuous compounded forward rates

Definition (Continuous Forward Rate)

Suppose a given observation time t and zero bond curve $P(t,\cdot):[t,\infty)\to\mathbb{R}^+$ for maturities $T\geq t$. The continuous compounded forward rate curve is given by

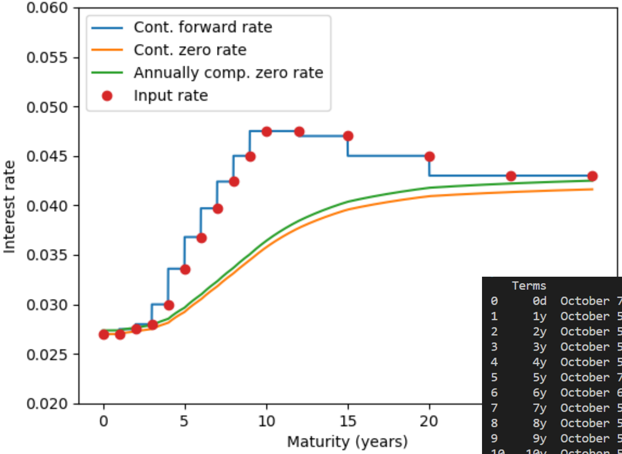
$$f(t,T) = -\frac{\partial \ln\left(P(t,T)\right)}{\partial T}.$$

From the definition follows

$$P(t,T) = \exp\left\{-\int_t^T f(t,s)ds\right\}.$$

- For static yield curve modelling and (simple) linear instrument pricing we are interested particularly in curves at $t=0$.
- For (more complex) option pricing we are interested in modelling curves at $t>0$.

We show a typical yield curve example



	Terms	Dates	Times	Rates
0	0d	October 7th, 2019	0	0.027
1	1y	October 5th, 2020	0.99726	0.027
2	2y	October 5th, 2021	1.99726	0.0275
3	3y	October 5th, 2022	2.99726	0.028
4	4y	October 5th, 2023	3.99726	0.03
5	5y	October 7th, 2024	5.00548	0.0336
6	6y	October 6th, 2025	6.00274	0.0368
7	7y	October 5th, 2026	7	0.0397
8	8y	October 5th, 2027	8	0.0424
9	9y	October 5th, 2028	9.00274	0.045
10	10y	October 5th, 2029	10.0027	0.0475
11	12y	October 6th, 2031	12.0055	0.0475
12	15y	October 5th, 2034	15.0055	0.047
13	20y	October 5th, 2039	20.0082	0.045
14	25y	October 5th, 2044	25.0137	0.043
15	30y	October 5th, 2049	30.0164	0.043

The market data for curve calibration is quoted by market data providers

Euribor vs 6 mth

1 Yr	-0.226/-0.266	16Yrs	1.295/1.255
2 Yrs	-0.128/-0.168	17Yrs	1.334/1.294
3 Yrs	0.010/-0.030	18Yrs	1.367/1.327
4 Yrs	0.154/0.114	19Yrs	1.393/1.353
5 Yrs	0.293/0.253	20Yrs	1.415/1.375
6 Yrs	0.429/0.389		
7 Yrs	0.558/0.518	21Yrs	1.432/1.392
8 Yrs	0.678/0.638	22Yrs	1.446/1.406
9 Yrs	0.790/0.750	23Yrs	1.457/1.417
10Yrs	0.892/0.852	24Yrs	1.465/1.425
		25Yrs	1.471/1.431
11Yrs	0.983/0.943		
12Yrs	1.064/1.024	26Yrs	1.476/1.436
13Yrs	1.135/1.095	27Yrs	1.480/1.440
14Yrs	1.197/1.157	28Yrs	1.484/1.444
15Yrs	1.250/1.210	29Yrs	1.486/1.446
		30Yrs	1.488/1.448
		35Yrs	1.491/1.451
		40Yrs	1.486/1.446
		45Yrs	1.476/1.436
		50Yrs	1.466/1.426
Disclaimer <IDIS>		60Yrs	1.456/1.416

3/6 basis
Spot Starting Date

1 Yr	4.30
2 Yrs	4.80
3 Yrs	5.35
4 Yrs	5.90
5 Yrs	6.40
6 Yrs	6.70
7 Yrs	6.85
8 Yrs	6.90
9 Yrs	6.90
10Yrs	6.85

Swap Spreads
(Gadget)

5y	59.3
10y	66.0

Page live in
London hours ONLY
(between 0700 - 1800)

This page will close 30th April
6.00pm and re open 7.00am 2nd May

10X12	0.192/0.152
10X15	0.378/0.338
10X20	0.543/0.503
10X25	0.599/0.559
10X30	0.616/0.576
10X35	0.619/0.579
10X40	0.614/0.574
10X45	0.604/0.564
10X50	0.594/0.554
10X60	0.584/0.544

Outline

Recall the introductory swap example

Interbank swap deal example

Pays 3% on 100mm EUR

Start date: Oct 30, 2020

End date: Oct 30, 2040

(annually, 30/360 day count, modified following, Target calendar)

Dates

Market conventions

Pays 6-months Euribor floating rate on 100mm EUR

Start date: Oct 30, 2020

End date: Oct 30, 2040

(semi-annually, act/360 day count, modified following, Target calendar)

How do we get from description to cash flow stream?

There are a couple of market conventions that need to be taken into account in practice

- Holiday calendars define at which dates payments can be made.
- Business day conventions specify how dates are adjusted if they fall on a non-business day.
- Schedule generation rules specify how regular dates are calculated.
- Day count conventions define how time is meassured between dates.

Outline

Dates are represented as triples day/month/year or as serial numbers

	A	B	C	D	E
1					
2					
3		**Date**	**Serial**	**EUR Payment System (TARGET)**	**London Bank Holiday**
4		Friday, July 27, 2018	43308	FALSE	FALSE
5		Monday, August 27, 2018	43339	FALSE	TRUE
6		Thursday, September 27, 2018	43370	FALSE	FALSE
7		Saturday, October 27, 2018	43400	TRUE	TRUE
8		Tuesday, November 27, 2018	43431	FALSE	FALSE
9		Thursday, December 27, 2018	43461	FALSE	FALSE
10		Sunday, January 27, 2019	43492	TRUE	TRUE
11		Wednesday, February 27, 2019	43523	FALSE	FALSE
12		Wednesday, March 27, 2019	43551	FALSE	FALSE
13		Saturday, April 27, 2019	43582	TRUE	TRUE
14		Monday, May 27, 2019	43612	FALSE	TRUE
15					
16		Sunday, January 1, 1900	1		
17					

A calender specifies business days and non-business days

Holiday Calendar

A holiday calendar $\mathcal{C}$ is a set of dates which are defined as holidays or non-business days.

- A particular date d is a non-business day if $d \in \mathcal{C}$.
- Holiday calendars are specific to a region, country or market segment.
- Need to be specified in the context of financial product.
- Typically contain weekends and special days of the year.
- May be joined (e.g. for multi-currency products), $\bar{\mathcal{C}} = \mathcal{C}_1 \cup \mathcal{C}_2$.
- Typical examples are TARGET calendar and LONDON calendar.

Outline

A business day convention maps non-business days to adjacent business days

Business Day Convention (BDC)

- A business day convention is a function $\omega_{\mathcal{C}} : \mathcal{D} \to \mathcal{D}$ which maps a date $d \in \mathcal{D}$ to another date $\bar{d}$.
- It is applied in conjunction with a calendar $\mathcal{C}$.
- Good business days are unchanged, i.e. $\omega_{\mathcal{C}}(d) = d$ if $d \notin \mathcal{C}$.

Following

$$\omega_{\mathcal{C}}(d) = \min\left\{\bar{d} \in \mathcal{D}\backslash\mathcal{C} \mid \bar{d} \geq d\right\}$$

Preceding

$$\omega_{\mathcal{C}}(d) = \max\left\{\bar{d} \in \mathcal{D}\backslash\mathcal{C} \mid \bar{d} \leq d\right\}$$

Modified Following

$$\omega_{\mathcal{C}}(d) = \begin{cases} \omega_{\mathcal{C}}^{\text{Following}}(d), & \text{if Month}\,[d] = \text{Month}\left[\omega_{\mathcal{C}}^{\text{Following}}(d)\right] \\ \omega_{\mathcal{C}}^{\text{Preceeding}}(d), & \text{else} \end{cases}$$

Outline

Schedules represent sets of regular reference dates

	Annual Frequency	TARGET Calendar	Modified Following
Start	Fri, 30 Oct 2020	FALSE	Fri, 30 Oct 2020
	Sat, 30 Oct 2021	TRUE	Fri, 29 Oct 2021
	Sun, 30 Oct 2022	TRUE	Mon, 31 Oct 2022
	Mon, 30 Oct 2023	FALSE	Mon, 30 Oct 2023
	Wed, 30 Oct 2024	FALSE	Wed, 30 Oct 2024
	Thu, 30 Oct 2025	FALSE	Thu, 30 Oct 2025
	Fri, 30 Oct 2026	FALSE	Fri, 30 Oct 2026
	Sat, 30 Oct 2027	TRUE	Fri, 29 Oct 2027
	Mon, 30 Oct 2028	FALSE	Mon, 30 Oct 2028
	Tue, 30 Oct 2029	FALSE	Tue, 30 Oct 2029
	Wed, 30 Oct 2030	FALSE	Wed, 30 Oct 2030
	Thu, 30 Oct 2031	FALSE	Thu, 30 Oct 2031
	Sat, 30 Oct 2032	TRUE	Fri, 29 Oct 2032
	Sun, 30 Oct 2033	TRUE	Mon, 31 Oct 2033
	Mon, 30 Oct 2034	FALSE	Mon, 30 Oct 2034
	Tue, 30 Oct 2035	FALSE	Tue, 30 Oct 2035
	Thu, 30 Oct 2036	FALSE	Thu, 30 Oct 2036
	Fri, 30 Oct 2037	FALSE	Fri, 30 Oct 2037
	Sat, 30 Oct 2038	TRUE	Fri, 29 Oct 2038
	Sun, 30 Oct 2039	TRUE	Mon, 31 Oct 2039
End	Tue, 30 Oct 2040	FALSE	Tue, 30 Oct 2040

Schedule generation follows some rules/conventions as well

1. Consider direction of roll-out: forward or backward (relevant for front/back stubs).
   1.1 Forward, roll-out from start (or effective) date to end (or maturity) date
   1.2 Backward, roll-out from end (or maturity) date to start (or effective) date
2. Roll out unadjusted dates according to frequency or tenor, e.g. annual frequency or 3 month tenor
3. If first/last period is broken consider short stub or long stub.
   3.1 Short stub is an unregular last period smaller then tenor.
   3.2 Long stub is an unregular last period larger then tenor
4. Adjust unadjusted dates according to calendar and BDC.

Outline

Day count conventions map dates to times or year fractions

Day Count Convention

A day count convention is a function $\tau : \mathcal{D} \times \mathcal{D} \to \mathbb{R}$ which measures a time period between dates in terms of years.

We give some examples:

Act/365 Fixed Convention

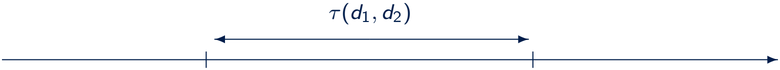
$\tau(d_1, d_2) = (d_2 - d_1)/365$

- Typically used to describe time in financial models.

Act/360 Convention

$\tau(d_1, d_2) = (d_2 - d_1)/360$

- Often used for Libor floating rate payments.



30/360 methods are slightly more involved

General 30/360 Method

- Consider two dates d_1 and d_2 represented as triples of day/month/year, i.e. $d_1 = [D_1, M_1, Y_1]$ and $d_2 = [D_2, M_2, Y_2]$ with $D_{1/2} \in \{1, \ldots, 31\}$, $M_{1/2} \in \{1, \ldots, 12\}$ and $Y_{1/2} \in \{1, 2, \ldots\}$.
- Obviously, only valid dates are allowed (no Feb. 30 or similar).
- Adjust $D_1 \mapsto \bar{D}_1$ and $D_2 \mapsto \bar{D}_2$ according to specific rules.
- Calculate

$$\tau(d_1, d_2) = \frac{360 \cdot (Y_2 - Y_1) + 30 \cdot (M_2 - M_1) + \left(\bar{D}_2 - \bar{D}_1\right)}{360}.$$

Some specific 30/360 rules are given below

30/360 Convention (or 30U/360, Bond Basis)

1. $\bar{D}_1 = \min\{D_1, 30\}$.
2. If $\bar{D}_1 = 30$ then $\bar{D}_2 = \min\{D_2, 30\}$ else if $\bar{D}_2 = D_2$.

30E/360 Convention (or Eurobond)

1. $\bar{D}_1 = \min\{D_1, 30\}$.
2. $\bar{D}_2 = \min\{D_2, 30\}$.

Outline

Now we have all pieces to price a deterministic coupon leg

Coupon is calculated as

$$\text{Coupon} = \text{Notional} \times \text{Rate} \times \text{YearFraction}$$
$$= 100,000,000\text{EUR} \times 3\% \times \tau$$

ValDate	Thu, 01 Oct 2020		Sum	41,787,559

	Annual Frequency	TARGET Calendar	Modified Following	_D1	_D2	tau	Rate	Coupon	P(0,T)	P(0,T)*Cpn
Start	Fri, 30 Oct 2020	FALSE	Fri, 30 Oct 2020							
	Sat, 30 Oct 2021	TRUE	Fri, 29 Oct 2021	30	29	0.997	3.00%	2,991,667	0.9713	2,905,943
	Sun, 30 Oct 2022	TRUE	Mon, 31 Oct 2022	29	31	1.006	3.00%	3,016,667	0.9451	2,850,916
	Mon, 30 Oct 2023	FALSE	Mon, 30 Oct 2023	30	30	1.000	3.00%	3,000,000	0.9192	2,757,657
	Wed, 30 Oct 2024	FALSE	Wed, 30 Oct 2024	30	30	1.000	3.00%	3,000,000	0.8927	2,678,166
	Thu, 30 Oct 2025	FALSE	Thu, 30 Oct 2025	30	30	1.000	3.00%	3,000,000	0.8646	2,593,664
	Fri, 30 Oct 2026	FALSE	Fri, 30 Oct 2026	30	30	1.000	3.00%	3,000,000	0.8345	2,503,445
	Sat, 30 Oct 2027	TRUE	Fri, 29 Oct 2027	30	29	0.997	3.00%	2,991,667	0.8031	2,402,572
	Mon, 30 Oct 2028	FALSE	Mon, 30 Oct 2028	29	30	1.003	3.00%	3,008,333	0.7704	2,317,730
	Tue, 30 Oct 2029	FALSE	Tue, 30 Oct 2029	30	30	1.000	3.00%	3,000,000	0.7373	2,211,969
	Wed, 30 Oct 2030	FALSE	Wed, 30 Oct 2030	30	30	1.000	3.00%	3,000,000	0.7039	2,111,644
	Thu, 30 Oct 2031	FALSE	Thu, 30 Oct 2031	30	30	1.000	3.00%	3,000,000	0.6713	2,013,762
	Sat, 30 Oct 2032	TRUE	Fri, 29 Oct 2032	30	29	0.997	3.00%	2,991,667	0.6401	1,915,033
	Sun, 30 Oct 2033	TRUE	Mon, 31 Oct 2033	29	31	1.006	3.00%	3,016,667	0.6103	1,841,155
	Mon, 30 Oct 2034	FALSE	Mon, 30 Oct 2034	30	30	1.000	3.00%	3,000,000	0.5822	1,746,731
	Tue, 30 Oct 2035	FALSE	Tue, 30 Oct 2035	30	30	1.000	3.00%	3,000,000	0.5555	1,666,418
	Thu, 30 Oct 2036	FALSE	Thu, 30 Oct 2036	30	30	1.000	3.00%	3,000,000	0.5300	1,590,074
	Fri, 30 Oct 2037	FALSE	Fri, 30 Oct 2037	30	30	1.000	3.00%	3,000,000	0.5060	1,518,029
	Sat, 30 Oct 2038	TRUE	Fri, 29 Oct 2038	30	29	0.997	3.00%	2,991,667	0.4833	1,445,981
	Sun, 30 Oct 2039	TRUE	Mon, 31 Oct 2039	29	31	1.006	3.00%	3,016,667	0.4617	1,392,766
End	Tue, 30 Oct 2040	FALSE	Tue, 30 Oct 2040	30	30	1.000	3.00%	3,000,000	0.4413	1,323,902

Outline

Outline

Recall the introductory swap example

Pays 3% on 100mm EUR

Start date: Oct 30, 2020

End date: Oct 30, 2040

(annually, 30/360 day count, modified following, Target calendar)

Stochastic interest rates

Pays 6-months Euribor floating rate on 100mm EUR

Start date: Oct 30, 2020

End date: Oct 30, 2040

(semi-annually, act/360 day count, modified following, Target calendar)

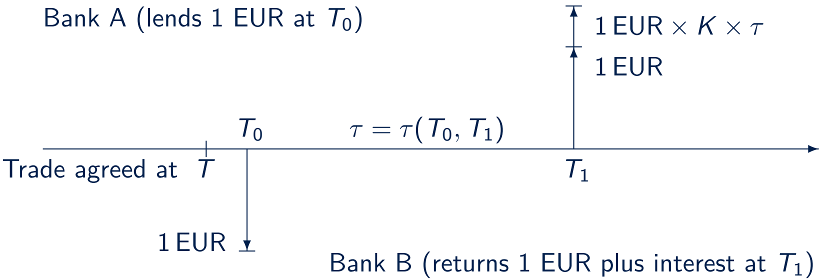
How do we model floating rates?

We start with some introductory remarks

- London Interbank Offered Rates (Libor) used to be the key building blocks of interest rate derivatives (for USD, GBP, JPY, CHF).
- EUR equivalent rate is Euribor rate - we will use Libor synonymously for Euribor.
- Libor rate modelling has undergone significant changes since financial crisis in 2008.
- This is typically reflected by the term Multi-Curve Interest Rate Modelling.
- Recent developments in the market lead to a shift away from Libor rates to alternative reference rates (Ibor Transition or Benchmark Reform).
- Alternative rates specifications lead to overnight index swaps.

Let's start with the classical Libor rate model

What is the fair interest rate K bank A and Bank B can agree on?



We get (via DCF methodology)

$$
\begin{aligned}
0 = V(T) &= P(T, T_0) \cdot \mathbb{E}^{T_0}\left[-1 \mid \mathcal{F}_T\right] + P(T, T_1) \cdot \mathbb{E}^{T_1}\left[1 + \tau K \mid \mathcal{F}_T\right], \\
0 &= -P(T, T_0) + P(T, T_1) \cdot (1 + \tau K).
\end{aligned}
$$

Spot Libor rates are fixed daily and quoted in the market

$$0 = -P(T, T_0) + P(T, T_1) \cdot (1 + \tau K)$$

Spot Libor rate

The fair rate for an interbank lending deal with trade date T, spot starting date T_0 (typically 0d or 2d after T) and maturity date T_1 is

$$L(T; T_0, T_1) = \left[\frac{P(T, T_0)}{P(T, T_1)} - 1 \right] \frac{1}{\tau}.$$

- Panel banks submit daily estimates for interbank lending rates to calculation agent.
- Relevant periods (i.e. $[T_0, T_1]$) considered are 1m, 3m, 6m and 12m.
- Trimmed average of submissions is calculated and published.

Libor rate fixings used to be the most important reference rates for interest rate derivatives. Nowadays, overnight rates become the key reference rates.

Example publication at Intercontinental Exchange (ICE) and EMMI

theice.com/marketdata/reports/170

ICE LIBOR Historical Rates

TENOR	PUBLICATION TIME*	USD ICE LIBOR 06-SEP-2018
Overnight	11:55:04 AM	1.91838
1 Week	11:55:04 AM	1.96100
1 Month	11:55:04 AM	2.13256
2 Month	11:55:04 AM	2.20950
3 Month	11:55:04 AM	2.32706
6 Month	11:55:04 AM	2.54419
1 Year	11:55:04 AM	2.84906

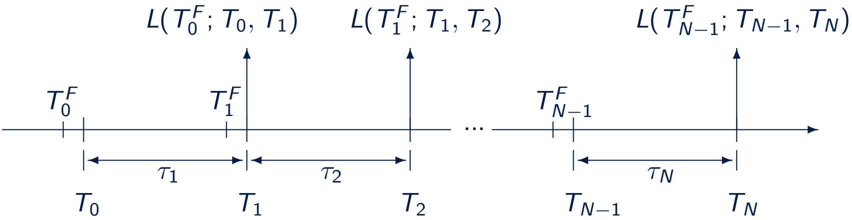
https://www.emmi-benchmarks.eu/benchmarks/

SS

Euribor

Date	1 Week	1 Month	3 Months	6 Months	12 Months
19 Apr 2022	-0.572	-0.560	-0.468	-0.333	-0.010

A plain vanilla Libor leg pays periodic Libor rate coupons



We get (via DCF methodology)

$$V(t) = \sum_{i=1}^{N} P(t, T_i) \cdot \mathbb{E}^{T_i}\left[L(T_{i-1}^F; T_{i-1}, T_i) \cdot \tau_i \mid \mathcal{F}_t\right]$$

$$= \sum_{i=1}^{N} P(t, T_i) \cdot \mathbb{E}^{T_i}\left[L(T_{i-1}^F; T_{i-1}, T_i) \mid \mathcal{F}_t\right] \cdot \tau_i .$$

Thus all we need is

$$\mathbb{E}^{T_i}\left[L(T_{i-1}^F; T_{i-1}, T_i) \mid \mathcal{F}_t\right] = ?$$

Libor rate is a martingale in the terminal measure (1/2)

Theorem (Martingale property of Libor rate)

The Libor rate $L(T; T_0, T_1)$ with observation/fixing date T, accrual start date T_0 and accrual end date T_1 is a martingale in the T_1-forward measure and

$$\mathbb{E}^{T_1}\left[L(T; T_0, T_1) \mid \mathcal{F}_t\right] = \left[\frac{P(t, T_0)}{P(t, T_1)} - 1\right]\frac{1}{\tau} = L(t; T_0, T_1).$$

Libor rate is a martingale in the terminal measure (2/2)

Proof.

Fair Libor rate at fixing time T is $L(T; T_0, T_1) = [P(T, T_0)/P(T, T_1) - 1]/\tau$. The zero coupon bond $P(T, T_0)$ is an asset and $P(T, T_1)$ is the numeraire in the T_1-forward meassure. Thus FTAP yields that the discounted asset price is a martingale, i.e.

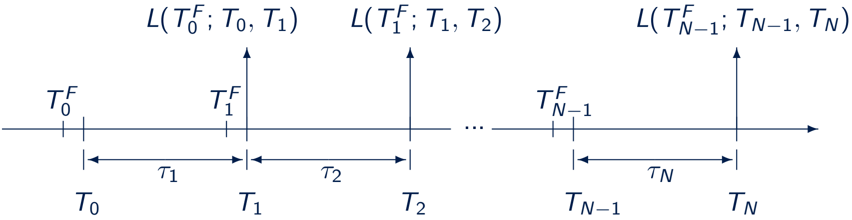
$$\mathbb{E}^{T_1}\left[\frac{P(T, T_0)}{P(T, T_1)} \mid \mathcal{F}_t\right] = \frac{P(t, T_0)}{P(t, T_1)}.$$

Linearity of expectation operator yields

$$\begin{aligned}
\mathbb{E}^{T_1}\left[L(T; T_0, T_1) \mid \mathcal{F}_t\right] &= \left[\mathbb{E}^{T_1}\left[\frac{P(T, T_0)}{P(T, T_1)} | \mathcal{F}_t\right] - 1\right]\frac{1}{\tau} \\
&= \left[\frac{P(t, T_0)}{P(t, T_1)} - 1\right]\frac{1}{\tau} \\
&= L(t; T_0, T_1).
\end{aligned}$$

□

This allows pricing the Libor leg based on today's knowledge of the yield curve only



Libor leg becomes

$$
\begin{aligned}
V(t) &= \sum_{i=1}^{N} P(t, T_i) \cdot \mathbb{E}^{T_i} \left[L(T_{i-1}^F; T_{i-1}, T_i) \cdot \tau_i \mid \mathcal{F}_t \right] \\
&= \sum_{i=1}^{N} P(t, T_i) \cdot L(t; T_{i-1}, T_i) \cdot \tau_i
\end{aligned}
$$

Libor leg may be simplified in the current single-curve setting

We have

$$V(t) = \sum_{i=1}^{N} P(t, T_i) \cdot L(t; T_{i-1}, T_i) \cdot \tau_i$$

with

$$L(t; T_{i-1}, T_i) = \left[\frac{P(t, T_{i-1})}{P(t, T_i)} - 1 \right] \frac{1}{\tau_i}.$$

This yields

$$\begin{aligned} V(t) &= \sum_{i=1}^{N} P(t, T_i) \cdot \left[\frac{P(t, T_{i-1})}{P(t, T_i)} - 1 \right] \frac{1}{\tau_i} \cdot \tau_i \\ &= \sum_{i=1}^{N} P(t, T_{i-1}) - P(t, T_i) \\ &= P(t, T_0) - P(t, T_N). \end{aligned}$$

We only need discount fators $P(t, T_0)$ and $P(t, T_N)$ at first date T_0 and last date T_N.

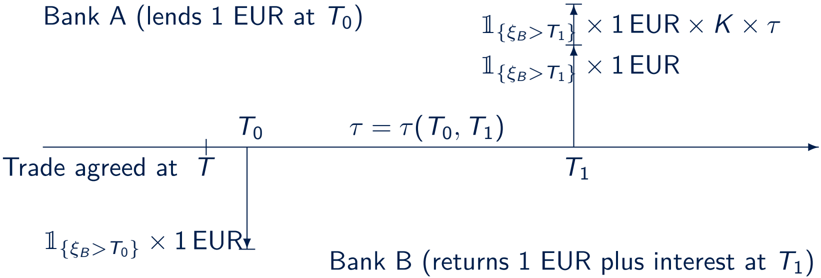
Outline

The classical Libor rate model misses an important detail

What if a counterparty defaults?

What if Bank B defaults prior to T_0 or T_1?

What is the fair rate K bank A and Bank B can agree on given the risk of default?



- Cash flows are paid only if no default occurs.
- We apply a simple credit model.
- Denote $\mathbb{1}_D$ the indicator function for an event D and random variable ξ_B the first time bank B defaults.

Credit-risky trade value can be derived using derivative pricing formula

$$\frac{V(T)}{B(T)} = \mathbb{E}^{\mathbb{Q}}\left[-\mathbb{1}_{\{\xi_B > T_0\}} \cdot \frac{1}{B(T_0)} + \mathbb{1}_{\{\xi_B > T_1\}} \cdot \frac{1 + K \cdot \tau}{B(T_1)}\right].$$

(all expectations conditional on $\mathcal{F}_T$)

Assume independence of credit event $\{\xi_B > T_{0/1}\}$ and interest rate market, then

$$\frac{V(T)}{B(T)} = -\mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{\{\xi_B > T_0\}}\right] \cdot \mathbb{E}^{\mathbb{Q}}\left[\frac{1}{B(T_0)}\right] + \mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{\{\xi_B > T_1\}}\right] \cdot \mathbb{E}^{\mathbb{Q}}\left[\frac{1 + K \cdot \tau}{B(T_1)}\right].$$

Abbreviate survival probability $Q(T, T_{0,1}) = \mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{\{\xi_B > T_{0,1}\}} \mid \mathcal{F}_T\right]$ and apply change of measure

$$V(T) = -P(T, T_0)Q(T, T_0)\mathbb{E}^{T_0}[1] + P(T, T_1)Q(T, T_1)\mathbb{E}^{T_1}[1 + K \cdot \tau].$$

This yields the fair spot rate in the presence of credit risk

$$V(T) = -P(T, T_0)Q(T, T_0)\mathbb{E}^{T_0}[1] + P(T, T_1)Q(T, T_1)\mathbb{E}^{T_1}[1 + K \cdot \tau].$$

If we solve $V(T) = 0$ and set $K = L(T; T_0, T_1)$ we get

$$L(T; T_0, T_1) = \left[\frac{P(T, T_0)}{P(T, T_1)} \cdot \frac{Q(T, T_0)}{Q(T, T_1)} - 1 \right] \frac{1}{\tau}.$$

We need a model for the survival probability $Q(T, T_{1,2})$.

Consider, e.g., hazard rate model $Q(T, T_{1,2}) = \exp\left\{ -\int_T^{T_{1,2}} \lambda(s) ds \right\}$ with deterministic hazard rate $\lambda(s)$. Then basis factor $D(T_0, T_1)$ with

$$D(T_0, T_1) = \frac{Q(T, T_0)}{Q(T, T_1)} = \exp\left\{ -\int_{T_0}^{T_1} \lambda(s) ds \right\}$$

is independent of observation time T.

Deterministic hazard rate assumption preserves the martingale property of forward Libor rate

Theorem (Martingale property of credit-risky Libor rate)

Consider the credit-risky Libor rate $L(T; T_0, T_1)$ with observation/fixing date T, accrual start date T_0 and accrual end date T_1. If the basis factor $D(T_0, T_1)$ is deterministic such that

$$L(T; T_0, T_1) = \left[\frac{P(T, T_0)}{P(T, T_1)} \cdot D(T_0, T_1) - 1 \right] \frac{1}{\tau},$$

then $L(t; T_0, T_1)$ is a martingale in the T_1-forward measure and

$$\mathbb{E}^{T_1} \left[L(T; T_0, T_1) \mid \mathcal{F}_t \right] = L(t; T_0, T_1) = \left[\frac{P(t, T_0)}{P(t, T_1)} \cdot D(T_0, T_1) - 1 \right] \frac{1}{\tau}.$$

Proof.

Follows analogously to classical Libor rate martingale property. □

Outline

Forward Libor rates are typically parametrised via projection curve

- Hazard rate $\lambda(u)$ in $Q(T, T_{1,2}) = \exp\left\{-\int_T^{T_{1,2}} \lambda(u)du\right\}$ is often considered as a tenor basis spread $s(u)$.
- Survival probability $Q(T, T_{1,2})$ can be interpreted as discount factor.
- Suppose we know time-t survival probabilities $Q(t, \cdot)$ for a forward Libor rate $L(t, T_0, T_0 + \delta)$ with tenor δ (typically 1m, 3m, 6m or 12m). Then we define the projection curve

$$P^\delta(t, T) = P(t, T) \cdot Q(t, T).$$

- With projection curve $P^\delta(t, T)$ the forward Libor rate formula is analogous to the classical Libor rate formula, i.e.

$$L^\delta(t, T_0) = L(t; T_0, T_0 + \delta) = \left[\frac{P^\delta(t, T_0)}{P^\delta(t, T_1)} - 1\right]\frac{1}{\tau}.$$

This yields the multi-curve modelling framework consisting of discount curve $P(t, T)$ and tenor-dependent projection curves $P^\delta(t, T)$.

There is an alternative approach to introduce multi-curve modelling

Define forward Libor rate $L^{\delta}(t, T_0)$ for a tenor δ as

$$L^{\delta}(t, T_0) = \mathbb{E}^{T_1}\left[L(T; T_0, T_0 + \delta) \mid \mathcal{F}_t\right].$$

(Without any assumptions on default, survival probabilities etc.)

Postulate a projection curve parametrisation

$$L^{\delta}(t, T_0) = \left[\frac{P^{\delta}(t, T_0)}{P^{\delta}(t, T_1)} - 1\right]\frac{1}{\tau}.$$

- We will discuss calibration of projection curve $P^{\delta}(t, T)$ later.
- This approach alone suffices for linear products (e.g. Libor legs) and simple options.
- It does not specify any relation between projection curve $P^{\delta}(t, T)$ and discount curve $P(t, T)$.

Projection curves can also be written in terms of zero rates and continuous forward rates

Consider a projection curve given by (pseudo) discount factors $P^{\delta}(t, T)$ (observed today).

- Corresponding continuous compounded zero rates are

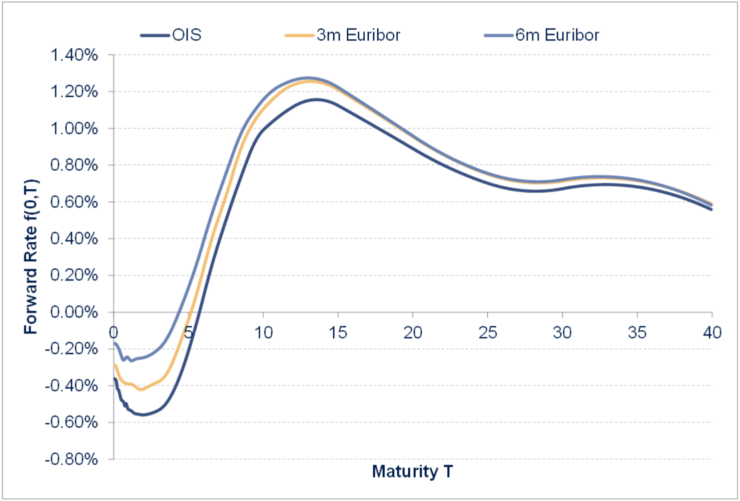
$$z^{\delta}(t, T) = -\frac{\ln\left[P^{\delta}(t, T)\right]}{T - t}.$$

- Corresponding continuous compounded forward rates are

$$f^{\delta}(t, T) = -\frac{\partial \ln\left[P^{\delta}(t, T)\right]}{\partial T}.$$

We illustrate an example of a multi-curve set-up for EUR

Market data as of July 2016



Libor leg pricing needs to be adapted slightly for multi-curve pricing

Classical single-curve Libor leg price is

$$\begin{aligned} V(t) &= \sum_{i=1}^{N} P(t, T_i) \cdot L(t; T_{i-1}, T_i) \cdot \tau_i \\ &= P(t, T_0) - P(t, T_N). \end{aligned}$$

Multi-curve Libor leg pricing becomes

$$V(t) = \sum_{i=1}^{N} P(t, T_i) \cdot L^{\delta}(t, T_{i-1}) \cdot \tau_i$$

with

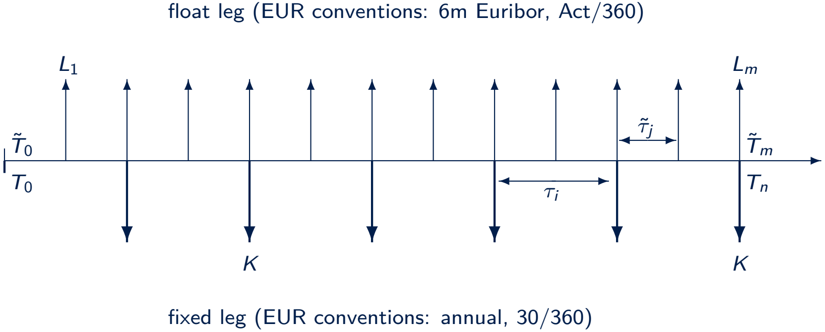
$$L^{\delta}(t, T_{i-1}) = \left[\frac{P^{\delta}(t, T_{i-1})}{P^{\delta}(t, T_i)} - 1 \right] \frac{1}{\tau_i}.$$

- Note that we need different yield curves for Libor rate projection and cash flow discounting.
- Single-curve pricing formula simplification does not work for multi-curve pricing.

Outline

Outline

With the fixed leg and Libor leg pricing available we can directly price a Vanilla interest rate swap



Present value of (fixed rate) payer swap with notional N becomes

$$V(t) = \sum_{j=1}^{m} N \cdot L^{6m}(t, \tilde{T}_{j-1}) \cdot \tilde{\tau}_j \cdot P(t, \tilde{T}_j) - \sum_{i=1}^{n} N \cdot K \cdot \tau_i \cdot P(t, T_i).$$

Vanilla swap pricing formula allows us to price the underlying swap of our introductory example

Interbank swap deal example

Pays 3% on 100mm EUR

Start date: Oct 30, 2020

End date: Oct 30, 2040

(annually, 30/360 day count, modified following, Target calendar)

Pays 6-months Euribor floating rate on 100mm EUR

Start date: Oct 30, 2020

End date: Oct 30, 2040

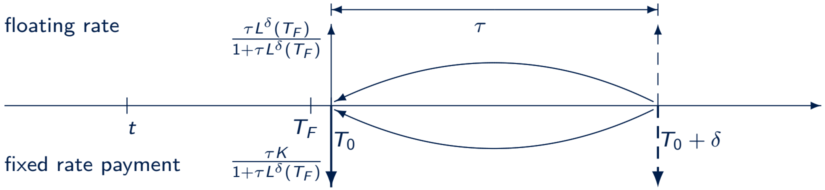
(semi-annually, act/360 day count, modified following, Target calendar)

We illustrate swap pricing with QuantLib/Excel...

- see YieldCurvesAndLegs.xlsx

Outline

Forward Rate Agreement yields exposure to single forward Libor rates



- Fixed rate K agreed at trade inception (prior to t).
- Libor rate $L^\delta(T_F, T_0)$ fixed at T_F, valid for the period T_0 to $T_0 + \delta$.
- Payoff paid at T_0 is difference $\tau \cdot \left[L^\delta(T_F, T_0) - K\right]$ discounted from T_1 to T_0 with discount factor $\left[1 + \tau \cdot L^\delta(T_F, T_0)\right]^{-1}$, i.e.

$$V(T_0) = \frac{\tau \cdot \left[L^\delta(T_F, T_0) - K\right]}{1 + \tau \cdot L^\delta(T_F, T_0)}.$$

Time-T_F FRA price can be obtained via deterministic basis spread model

Note that payoff $V(T_0) = \frac{\tau \cdot \left[L^\delta(T_F, T_0) - K\right]}{1+\tau \cdot L^\delta(T_F, T_0)}$ is already determined at T_F. Thus (via DCF)

$$V(T_F) = P(T_F, T_0) \cdot V(T_0) = P(T_F, T_0) \cdot \frac{\tau \cdot \left[L^\delta(T_F, T_0) - K\right]}{1 + \tau \cdot L^\delta(T_F, T_0)}.$$

Recall that (with $T_1 = T_0 + \delta$)

$$1 + \tau \cdot L^\delta(T_F, T_0) = \frac{P^\delta(T_F, T_0)}{P^\delta(T_F, T_1)} = \frac{P(T_F, T_0)}{P(T_F, T_1)} \cdot D(T_0, T_1).$$

Then

$$\begin{aligned} V(T_F) &= P(T_F, T_0) \cdot \tau \cdot \left[L^\delta(T_F, T_0) - K\right] \cdot \frac{1}{D(T_0, T_1)} \cdot \frac{P(T_F, T_1)}{P(T_F, T_0)} \\ &= P(T_F, T_1) \cdot \tau \cdot \left[L^\delta(T_F, T_0) - K\right] \cdot \frac{1}{D(T_0, T_1)}. \end{aligned}$$

Present value of FRA can be obtained via martingale property

Derivative pricing formula in T_1-terminal measure yields

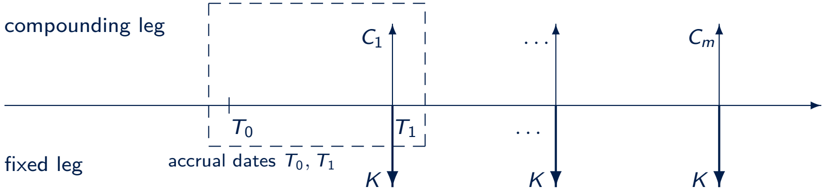
$$\frac{V(t)}{P(t,T_1)} = \mathbb{E}^{T_1}\left[\frac{P(T_F,T_1)}{P(T_F,T_1)}\cdot\tau\cdot\left[L^\delta(T_F,T_0)-K\right]\cdot\frac{1}{D(T_0,T_1)}\right]$$
$$= \tau\cdot\left[\mathbb{E}^{T_1}\left[L^\delta(T_F,T_0)\right]-K\right]\cdot\frac{1}{D(T_0,T_1)}$$
$$= \tau\cdot\left[L^\delta(t,T_0)-K\right]\cdot\frac{1}{D(T_0,T_1)}.$$

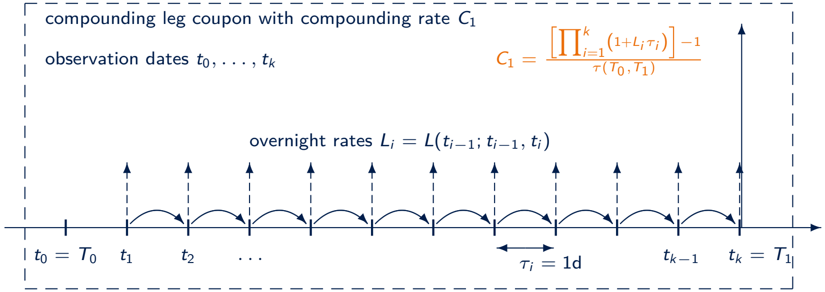
Using $1+\tau\cdot L^\delta(t,T_0)=\frac{P(t,T_0)}{P(t,T_1)}\cdot D(T_0,T_1)$ (deterministic spread assumption) yields

$$V(t) = P(t,T_0)\cdot\tau\cdot\left[L^\delta(t,T_0)-K\right]\cdot\left[\frac{P(t,T_0)}{P(t,T_1)}\cdot D(T_0,T_1)\right]^{-1}$$
$$= P(t,T_0)\cdot\frac{\left[L^\delta(t,T_0)-K\right]\cdot\tau}{1+\tau\cdot L^\delta(t,T_0)}.$$

Outline

Overnight index swap (OIS) instruments are further relevant instruments in the market





We need to calculate the compounding leg coupon rate

- Assume overnight rate $L_i = L(t_{i-1}; t_{i-1}, t_i)$ is a credit-risk free Libor rate. In practice often simply called risk-free rate (RFR)
- Compounded rate (for a period $[T_0, T_1]$) is specified as

$$C_1 = \left\{ \left[\prod_{i=1}^{k} (1 + L_i \tau_i) \right] - 1 \right\} \frac{1}{\tau(T_0, T_1)}.$$

- Coupon payment is at T_1.
- For pricing we need to calculate

$$\begin{aligned} \mathbb{E}^{T_1}\left[C_1 \,|\, \mathcal{F}_t\right] &= \mathbb{E}^{T_1}\left[\left\{ \left[\prod_{i=1}^{k} (1 + L_i \tau_i) \right] - 1 \right\} \frac{1}{\tau(T_0, T_1)} \,|\, \mathcal{F}_t \right] \\ &= \left\{ \mathbb{E}^{T_1}\left[\prod_{i=1}^{k} (1 + L_i \tau_i) \,|\, \mathcal{F}_t \right] - 1 \right\} \frac{1}{\tau(T_0, T_1)}. \end{aligned}$$

How do we handle the compounding term?

Overall compounding term is

$$\prod_{i=1}^{k}(1+L_i\tau_i)=\prod_{i=1}^{k}\left[1+L(t_{i-1};t_{i-1},t_i)\tau_i\right].$$

Individual compounding term is

$$1+L(t_{i-1};t_{i-1},t_i)\tau_i=1+\left[\frac{P(t_{i-1},t_{i-1})}{P(t_{i-1},t_i)}-1\right]\frac{1}{\tau_i}\tau_i=\frac{P(t_{i-1},t_{i-1})}{P(t_{i-1},t_i)}.$$

We get

$$\prod_{i=1}^{k}(1+L_i\tau_i)=\prod_{i=1}^{k}\frac{P(t_{i-1},t_{i-1})}{P(t_{i-1},t_i)}=\prod_{i=1}^{k}\frac{1}{P(t_{i-1},t_i)}.$$

We need to calculate the expectation of $\prod_{i=1}^{k}\frac{1}{P(t_{i-1},t_i)}$.

Expected compounding factor can easily be calculated

Lemma (Compounding rate)

Consider a compounding coupon period $[T_0, T_1]$ *with overnight observation and maturity dates* $\{t_0, t_1, \ldots, t_k\}$, $t_0 = T_0$ *and* $t_k = T_1$. *Then*

$$\mathbb{E}^{T_1}\left[\prod_{i=1}^{k} \frac{1}{P(t_{i-1}, t_i)} \,|\, \mathcal{F}_{T_0}\right] = \frac{1}{P(T_0, T_1)}.$$

For the proof we use the notation $\mathbb{E}^{T_1}[\cdot \,|\, \mathcal{F}_t] = \mathbb{E}_t^{T_1}[\cdot]$.

We proof the result via Tower Law of conditional expectation

$$
\begin{aligned}
\mathbb{E}_{T_0}^{T_1}\left[\prod_{i=1}^{k}\frac{1}{P(t_{i-1},t_i)}\right] &= \mathbb{E}_{T_0}^{T_1}\left[\mathbb{E}_{t_{k-2}}^{T_1}\left[\prod_{i=1}^{k}\frac{1}{P(t_{i-1},t_i)}\right]\right] \\
&= \mathbb{E}_{T_0}^{T_1}\left[\prod_{i=1}^{k-1}\frac{1}{P(t_{i-1},t_i)}\,\mathbb{E}_{t_{k-2}}^{T_1}\left[\frac{P(t_{k-1},t_{k-1})}{P(t_{k-1},t_k)}\right]\right] \\
&= \mathbb{E}_{T_0}^{T_1}\left[\prod_{i=1}^{k-1}\frac{1}{P(t_{i-1},t_i)}\,\frac{P(t_{k-2},t_{k-1})}{P(t_{k-2},t_k)}\right] \\
&= \mathbb{E}_{T_0}^{T_1}\left[\prod_{i=1}^{k-2}\frac{1}{P(t_{i-1},t_i)}\,\frac{1}{P(t_{k-2},t_k)}\right] \\
\ldots &= \mathbb{E}_{T_0}^{T_1}\left[\frac{1}{P(t_0,t_k)}\right] \\
&= \frac{1}{P(T_0,T_1)}.
\end{aligned}
$$

Expected compounding rate equals Libor rate

- Expected compounding rate as seen at start date T_0 becomes

$$\mathbb{E}^{T_1}\left[C_1 \,|\, \mathcal{F}_{T_0}\right] = \left[\frac{1}{P(T_0, T_1)} - 1\right] \frac{1}{\tau(T_0, T_1)} = L(T_0; T_0, T_1).$$

- Consequently, expected compounding rate equals Libor rate for full period.

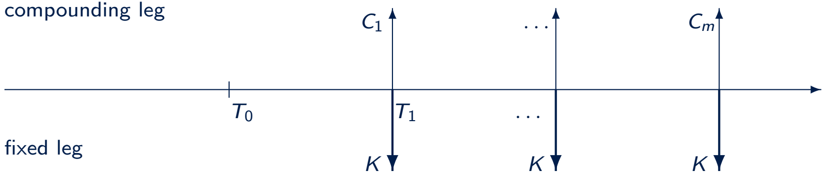
- Moreover, expectations as seen of time-t are

$$\mathbb{E}^{T_1}\left[\prod_{i=1}^{k} \frac{1}{P(t_{i-1}, t_i)} \,|\, \mathcal{F}_t\right] = \frac{P(t, T_0)}{P(t, T_1)}$$

and

$$\mathbb{E}^{T_1}\left[C_1 \,|\, \mathcal{F}_t\right] = \left[\frac{P(t, T_0)}{P(t, T_1)} - 1\right] \frac{1}{\tau(T_0, T_1)} = L(t; T_0, T_1).$$

Compounding swap pricing is analogous to Vanilla swap pricing



$$
\begin{aligned}
V(t) &= \sum_{j=1}^{m} N \cdot \mathbb{E}^{T_j}\left[C_j \,|\, \mathcal{F}_t\right] \cdot \tau_j \cdot P(t, T_j) - \sum_{j=1}^{m} N \cdot K \cdot \tau_j \cdot P(t, T_j) \\
&= \sum_{j=1}^{m} N \cdot L(t; T_{j-1}, T_j) \cdot \tau_j \cdot P(t, T_j) - \sum_{j=1}^{m} N \cdot K \cdot \tau_j \cdot P(t, T_j).
\end{aligned}
$$

Outline

As a summary we give an overview of linear products pricing

Vanilla (Payer) Swap

$$\text{Swap}(t) = \underbrace{\sum_{j=1}^{m} N \cdot L^{\delta}(t, \tilde{T}_{j-1}) \cdot \tilde{\tau}_j \cdot P(t, \tilde{T}_j)}_{\text{float leg}} - \underbrace{\sum_{i=1}^{n} N \cdot K \cdot \tau_i \cdot P(t, T_i)}_{\text{fixed Leg}}$$

Market Forward Rate Agreement (FRA)

$$\text{FRA}(t) = \underbrace{P(t, T_0)}_{\text{discounting to } T_0} \cdot \underbrace{\left[L^{\delta}(t, T_0) - K\right] \cdot \tau}_{\text{payoff}} \cdot \underbrace{\frac{1}{1 + \tau \cdot L^{\delta}(t, T_0)}}_{\text{discounting from } T_0 \text{ to } T_0+\delta}$$

Compounding Swap / OIS Swap

$$\text{CompSwap}(t) = \underbrace{\sum_{j=1}^{m} N \cdot L(t; T_{j-1}, T_j) \cdot \tau_j \cdot P(t, T_j)}_{\text{compounding leg}} - \underbrace{\sum_{j=1}^{m} N \cdot K \cdot \tau_j \cdot P(t, T_j)}_{\text{fixed leg}}$$

Further reading on yield curves, conventions and linear products

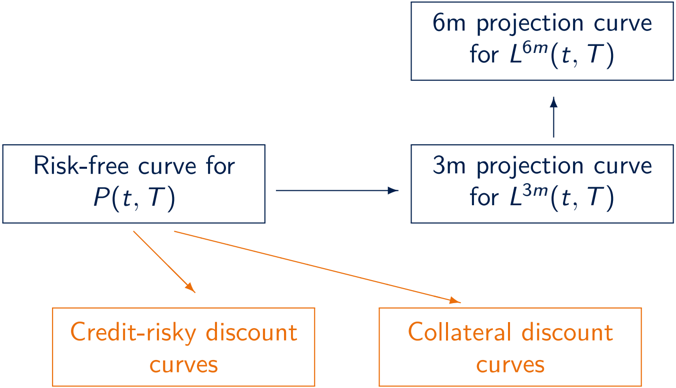
- F. Ametrano and M. Bianchetti. Everything you always wanted to know about Multiple Interest Rate Curve Bootstrapping but were afraid to ask (April 2, 2013).
  Available at SSRN: http://ssrn.com/abstract=2219548 or http://dx.doi.org/10.2139/ssrn.2219548, 2013
- M. Henrard. Interest rate instruments and market conventions guide 2.0.
  Open Gamma Quantitative Research, 2013
- P. Hagan and G. West. Interpolation methods for curve construction.
  Applied Mathematical Finance, 13(2):89–128, 2006

On current discussion of Libor alternatives, e.g.

- M. Henrard. A quant perspective on ibor fallback proposals.
  https://ssrn.com/abstract=3226183, 2018

Outline

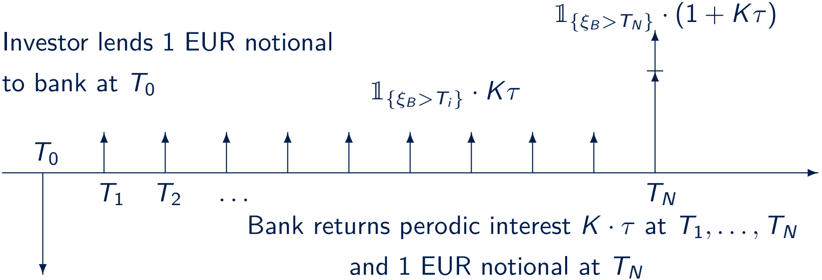
So far we discussed risk-free discount curves and tenor forward curves - now it is getting a bit more complex



Specifying appropriate discount and projection curves for a financial instrument is an important task in practice.

Outline

Discounting of bond or loan cash flows is subject to credit risk



- Cash flows are paid only if no default occurs.
- Denote $\mathbb{1}_D$ the indicator function for an event D and random variable ξ_B the first time bank defaults.
- Assume independence of credit event $\{\xi_B > T\}$ and interest rate market

We repeat credit-risky valuation from multi-curve pricing

Consider an observation time t with $T_0 < t \le T_N$ then present value of bond cash flows becomes

$$\frac{V(t)}{B(t)} = \mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{\{\xi_B > T_N\}}\frac{1}{B(T_N)} + \sum_{T_i \ge t}\mathbb{1}_{\{\xi_B > T_i\}}\frac{K\tau}{B(T_i)} \,|\, \mathcal{F}_t\right].$$

Independence of credit event $\{\xi_B > T\}$ and interest rate market yields (all expectations conditional on $\mathcal{F}_t$)

$$\frac{V(t)}{B(t)} = \mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{\{\xi_B > T_N\}}\right]\mathbb{E}^{\mathbb{Q}}\left[\frac{1}{B(T_N)}\right] + \sum_{T_i \ge t}\mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{\{\xi_B > T_i\}}\right]\mathbb{E}^{\mathbb{Q}}\left[\frac{K\tau}{B(T_i)}\right].$$

Denote survival probability $Q(t,T) = \mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{\{\xi_B > T\}} \,|\, \mathcal{F}_t\right]$ and change to forward measure, then

$$V(t) = Q(t,T_N)P(t,T_N) + \sum_{T_i \ge t} Q(t,T_i)P(t,T_i)K\tau.$$

Survival probabilities are parameterized in terms of spread curves - this leads to credit-risky discount curves

Assume survival probability $Q(t, T)$ is given in terms of a credit spread curve $s(t)$ and

$$Q(t, T) = \exp\left\{-\int_t^T s(u)du\right\}.$$

Also recall that discount factors may be represented in terms of forward rates $f(t, T)$ and

$$P(t, T) = \exp\left\{-\int_t^T f(t, u)du\right\}.$$

We may define a credit-risky discount curve $P^B(t, T)$ for a bond or loan as

$$P^B(t, T) = Q(t, T)P(t, T) = \exp\left\{-\int_t^T [f(t, u) + s(u)]\, du\right\}.$$

We can adapt the discounted cash flow pricing method to cash flows subject to credit risk

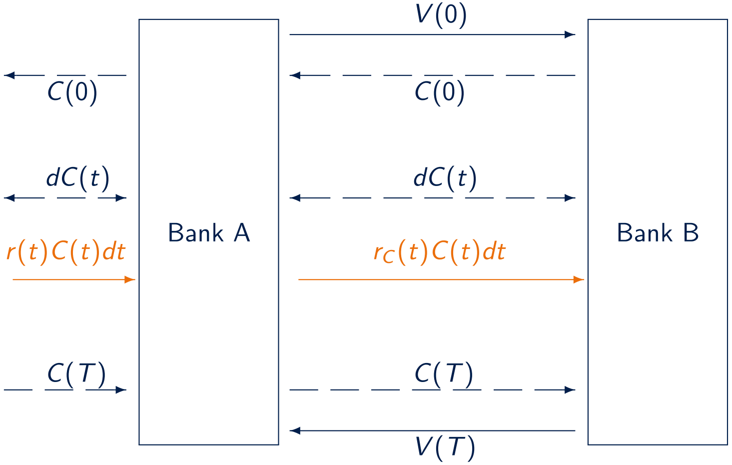
Present value of bond or loan cash flows become

$$V(t) = P^B(t, T_N) + \sum_{T_i \geq t} P^B(t, T_i) K \tau.$$

- Bonds are issued by many market participants (banks, corporates, governments, ...)
- Credit spread curves and credit-risky discount curves are specific to an issuer, e.g. Deutsche Bank has a different credit spread than Bundesrepublik Deutschland
- Many bonds are actively traded in the market. Then we may use market prices and infer credit spreads $s(t)$ and credit-risky discount curves $P^B(t, T)$

Outline

For derivative transactions credit risk is typically mitigated by posting collateral



Pricing needs to take into account interest payments on collateral.[2]

[2] Collateral amounts $C(t)$ and collateral rates are agreed in *Credit Support Annexes* (CSAs) between counterparties.

Collateralized derivative pricing takes into account collateral cash flows

Collateralized derivative price is given by (expectation of) sum of discounted payoff

$$e^{-\int_t^T r(u)du} V(T)$$

plus sum of discounted collateral interest payments

$$\int_t^T e^{-\int_t^s r(u)du} \left[r(s) - r_C(s) \right] C(s) ds.$$

That gives

$$V(t) = \mathbb{E}^{\mathbb{Q}} \left[e^{-\int_t^T r(u)du} V(T) + \int_t^T e^{-\int_t^s r(u)du} \left[r(s) - r_C(s) \right] C(s) ds \,|\, \mathcal{F}_t \right].$$

Pricing is reformulated to focus on collateral rate (1/2)

From

$$V(t) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(u)du} V(T) + \int_t^T e^{-\int_t^s r(u)du} \left[r(s) - r_C(s) \right] C(s) ds \,|\, \mathcal{F}_t \right]$$

we can derive:

Theorem (Collateralized Discounting)

Consider the price of an option $V(t)$ at time t which pays an amount $V(T)$ at time $T \geq t$ (and no intermediate cash flows).
The option is assumed collateralized with cash amounts $C(s)$ (for $t \leq s \leq T$). For the cash collateral a collateral rate $r_C(s)$ (for $t \leq s \leq T$) is applied.
Then the option price $V(t)$ becomes

$$V(t) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_C(u)du} V(T) \,|\, \mathcal{F}_t \right] - \mathbb{E}^{\mathbb{Q}}\left[\int_t^T e^{-\int_t^s r_C(u)du} \left[r(s) - r_C(s) \right] \left[V(s) - C(s) \right] ds \,|\, \mathcal{F}_t \right]$$

Pricing is reformulated to focus on collateral rate (2/2)

For further details on collateralized discounting see, e.g.

- V. Piterbarg. Funding beyond discounting: collateral agreements and derivatives pricing.
  Asia Risk, pages 97–102, February 2010

- M. Fujii, Y. Shimada, and A. Takahashi. Collateral posting and choice of collateral currency - implications for derivative pricing and risk management (may 8, 2010).
  Available at SSRN: https://ssrn.com/abstract=1601866, May 2010

Collateralized discounting result is proved in three steps

1. Define the discounted collateralized price process

$$X(t) = e^{-\int_0^t r(u)du} V(t) + \int_0^t e^{-\int_0^s r(u)du} \left[r(s) - r_C(s) \right] C(s) ds$$

   and show that it is a martingale

2. Analyse the dynamics $dX(t)$ and deduce the dynamics for $dV(t)$

3. Solve the SDE for $dV(t)$ and calculate price via conditional expectation

Step 1 - discounted collateralized price process (1/2)

Consider $T \geq t$, then

$$
\begin{aligned}
X(T) &= e^{-\int_0^T r(u)du} V(T) + \int_0^T e^{-\int_0^s r(u)du} \left[r(s) - r_C(s) \right] C(s) ds \\
&= e^{-\int_0^T r(u)du} V(T) + \int_0^t e^{-\int_0^s r(u)du} \left[r(s) - r_C(s) \right] C(s) ds + \\
&\quad \int_t^T e^{-\int_0^s r(u)du} \left[r(s) - r_C(s) \right] C(s) ds \\
&= e^{-\int_0^t r(u)du} \underbrace{\left[e^{-\int_t^T r(u)du} V(T) + \int_t^T e^{-\int_t^s r(u)du} \left[r(s) - r_C(s) \right] C(s) ds \right]}_{K(t,T)} + \\
&\quad \int_0^t e^{-\int_0^s r(u)du} \left[r(s) - r_C(s) \right] C(s) ds.
\end{aligned}
$$

Step 1 - discounted collateralized price process (2/2)

We have from collateralized derivative pricing that

$$\mathbb{E}^{\mathbb{Q}}\left[K(t,T)\,|\,\mathcal{F}_t\right] = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(u)du}V(T) + \int_t^T e^{-\int_t^s r(u)du}\left[r(s) - r_C(s)\right]C(s)ds\,|\,\mathcal{F}_t\right]$$
$$= V(t).$$

This yields

$$\mathbb{E}^{\mathbb{Q}}\left[X(T)\,|\,\mathcal{F}_t\right] = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^t r(u)du}K(t,T) + \int_0^t e^{-\int_0^s r(u)du}\left[r(s) - r_C(s)\right]C(s)ds\,|\,\mathcal{F}_t\right]$$
$$= e^{-\int_0^t r(u)du}\mathbb{E}^{\mathbb{Q}}\left[K(t,T)\,|\,\mathcal{F}_t\right] + \int_0^t e^{-\int_0^s r(u)du}\left[r(s) - r_C(s)\right]C(s)ds$$
$$= e^{-\int_0^t r(u)du}V(t) + \int_0^t e^{-\int_0^s r(u)du}\left[r(s) - r_C(s)\right]C(s)ds$$
$$= X(t).$$

Thus, $X(t)$ is indeed a martingale.

Step 2 - dynamics $dX(t)$ and $dV(t)$

From $X(t) = e^{-\int_0^t r(u)du} V(t) + \int_0^t e^{-\int_0^s r(u)du} [r(s) - r_C(s)] \, C(s)ds$ follows

$$
\begin{aligned}
dX(t) = & -r(t)e^{-\int_0^t r(u)du} V(t)dt + e^{-\int_0^t r(u)du} dV(t) + \\
& e^{-\int_0^t r(u)du} [r(t) - r_C(t)] \, C(t)dt \\
= & \, e^{-\int_0^t r(u)du} [dV(t) - r(t)V(t)dt + [r(t) - r_C(t)] \, C(t)dt] \\
= & \, e^{-\int_0^t r(u)du} \underbrace{[dV(t) - r_C(t)V(t)dt + [r(t) - r_C(t)] [C(t) - V(t)] \, dt]}_{dM(t)}.
\end{aligned}
$$

Since $X(t)$ is a martingale we must have that $dM(t)$ are increments of a martingale.
We get

$$dV(t) = r_C(t)V(t)dt - [r(t) - r_C(t)] [C(t) - V(t)] \, dt + dM(t).$$

Step 3 - solution for $V(t)$ (1/2)

For the SDE $dV(t) = r_C(t)V(t)dt - [r(t) - r_C(t)]\,[C(t) - V(t)]\,dt + dM(t)$ we may guess a solution as

$$V(t) = e^{\int_{t_0}^{t} r_C(s)ds} V(t_0) - \int_{t_0}^{t} e^{\int_s^t r_C(u)du} \left\{[r(s) - r_C(s)]\,[C(s) - V(s)]\,ds - dM(s)\right\}$$

Differentiating confirms that

$$\begin{aligned}
dV(t) &= r_C(t) e^{\int_{t_0}^{t} r_C(s)ds} V(t_0) \\
&\qquad - r_C(t) \int_{t_0}^{t} e^{\int_s^t r_C(u)du} \left\{[r(s) - r_C(s)]\,[C(s) - V(s)]\,ds - dM(s)\right\} \\
&\qquad - e^{\int_t^t r_C(u)du} \left\{[r(t) - r_C(t)]\,[C(t) - V(t)]\,dt - dM(t)\right\} \\
&= r_C(t) \Big[e^{\int_{t_0}^{t} r_C(s)ds} V(t_0) - \\
&\qquad \int_{t_0}^{t} e^{\int_s^t r_C(u)du} \left\{[r(s) - r_C(s)]\,[C(s) - V(s)]\,ds - dM(s)\right\} \Big] \\
&\qquad - [r(t) - r_C(t)]\,[C(t) - V(t)]\,dt + dM(t) \\
&= r_C(t)V(t) - [r(t) - r_C(t)]\,[C(t) - V(t)]\,dt + dM(t).
\end{aligned}$$

Step 3 - solution for $V(t)$ (2/2)

Substituting $t \mapsto T$ and $t_0 \mapsto t$ yields the representation

$$V(T) = e^{\int_t^T r_C(s)ds} V(t) - \int_t^T e^{\int_s^T r_C(u)du} \{[r(s) - r_C(s)]\,[C(s) - V(s)]\, ds - dM(s)\}$$

Solving for $V(t)$ gives

$$V(t) = e^{-\int_t^T r_C(s)ds} V(T) - \int_t^T e^{-\int_t^s r_C(u)du} \{[r(s) - r_C(s)]\,[V(s) - C(s)]\, ds - dM(s)\}$$

The result follows now from taking conditional expectation

$$V(t) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_C(s)ds} V(T) - \int_t^T e^{-\int_t^s r_C(u)du} [r(s) - r_C(s)]\,[V(s) - C(s)]\, ds \,\middle|\, \mathcal{F}_t \right] + \underbrace{\mathbb{E}^{\mathbb{Q}}\left[\int_t^T e^{-\int_t^s r_C(u)du} dM(s) \,\middle|\, \mathcal{F}_t \right]}_{0}$$

A very important special case arises for full collateralization

Corollary (Full collateralization)

If the collateral amount $C(s)$ equals the full option price $V(s)$ for $t \leq s \leq T$ then the derivative price becomes

$$V(t) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_C(s)ds} V(T) \,|\, \mathcal{F}_t\right].$$

- Fully collateralized price is calculated analogous to uncollateralized price.
- Discount rate must be equal to the collateral rate $r_C(s)$.
- Pricing is independent of the risk-free rate $r(t)$.
- Collateral bank account $B^C(t) = \exp\left\{\int_0^t r_C(s)ds\right\}$ can be considered as numeraire in this setting

The collateralized zero coupon bond can be used to adapt DCF method to collateralized derivative pricing

Consider a fully collateralized instrument that pays $V(T) = 1$ at some time horizon T. The price $V(t)$ for $t \leq T$ is given by
$V(t) = \mathbb{E}^{\mathbb{Q}} \left[e^{-\int_t^T r_C(s)ds} 1 \,|\, \mathcal{F}_t \right]$.

Definition (Collateralized zero coupon bond)

The collateralized zero coupon bond price (or collateralized discount factor) for an observation time t and maturity $T \geq t$ is given by

$$P^C(t,T) = \mathbb{E}^{\mathbb{Q}} \left[e^{-\int_t^T r_C(s)ds} \,|\, \mathcal{F}_t \right].$$

Consider a time horizon T and the time-t price process of a collateralized zero coupon bond $P^C(t,T)$:

- Collateralized zero coupon bond is an asset in our economy,
- price process $P^C(t,T) > 0$.

Thus collateralized zero coupon bond is a numeraire.

The collateralized zero coupon bond can be used as numeraire for pricing

Define the collateralized forward measure $\mathbb{Q}^{T,C}$ as the equivalent martingale measure with $P^C(t,T)$ as numeraire and expectation $\mathbb{E}^{T,C}[\cdot]$. The density process of $\mathbb{Q}^{T,C}$ (relative to risk-neutral measure $\mathbb{Q}$) is

$$\zeta(t) = \frac{P^C(t,T)}{B^C(t)} \cdot \frac{B^C(0)}{P^C(0,T)}.$$

This yields

$$\begin{aligned}
\mathbb{E}^{T,C}\left[V(T)\,|\,\mathcal{F}_t\right] &= \mathbb{E}^{\mathbb{Q}}\left[\frac{\zeta(T)}{\zeta(t)}V(T)\,|\,\mathcal{F}_t\right] \\
&= \mathbb{E}^{\mathbb{Q}}\left[\frac{P^C(T,T)}{B^C(T)} \cdot \frac{B^C(t)}{P^C(t,T)}V(T)\,|\,\mathcal{F}_t\right] \\
&= \frac{1}{P^C(t,T)}\mathbb{E}^{\mathbb{Q}}\left[\frac{B^C(t)}{B^C(T)} \cdot V(T)\,|\,\mathcal{F}_t\right] \\
&= \frac{1}{P^C(t,T)}\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_C(s)ds}V(T)\,|\,\mathcal{F}_t\right] = \frac{V(t)}{P^C(t,T)}.
\end{aligned}$$

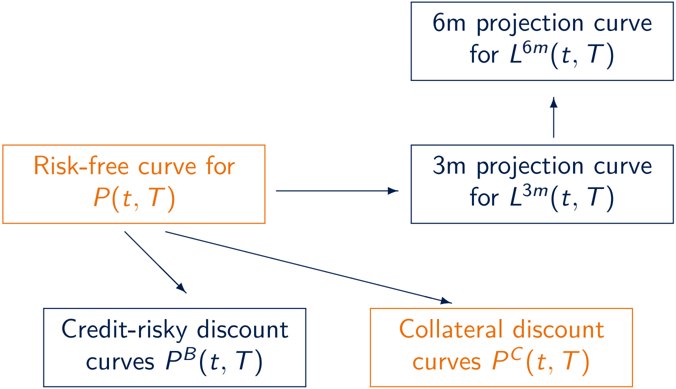
Discounted cash flow method pricing requires to use the appropriate discount curve representing collateral rates

We have

$$V(t) = P^C(t, T) \cdot \mathbb{E}^{T,C}\left[V(T) \,|\, \mathcal{F}_t\right].$$

- Requires discounting curve $P^C(t, T) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_C(s)ds} \,|\, \mathcal{F}_t\right]$ capturing collateral costs and
- calculation of expected future payoffs $\mathbb{E}^{T,C}\left[V(T) \,|\, \mathcal{F}_t\right]$ in the collateralized forward measure.

We summarise the multi-curve framework widely adopted in the market



- Standard collateral curve is also considered as risk-free curve.
- In 2020 standard collateral curves move to €STR collateral rate (EUR) and SOFR collateral rate (USD).
- Projection curves are potentially not required anymore in the future if Libor (and Euribor) indices are decommissioned.

Outline

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