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Interest Rate Modelling and Derivative Pricing, Summer Term 2025

Exercise 3

1. **Black-Formula for log-normal model**
   Consider the lognormal model
   $$dS(t) = \sigma \cdot S(t) \cdot dW(t)$$
   with constant volatility σ.

   (a) Show that the forward price of a call option is given by
   $$\mathbb{E}\left[[S(T) - K]^+ \mid \mathcal{F}_t\right] = \text{Black}\left(S(t), K, \sigma\sqrt{T-t}, 1\right)$$
   with
   $$\text{Black}\left(F, K, \nu, \phi\right) = \phi \cdot \left[F \cdot \Phi\left(\phi \cdot d_1\right) - K \cdot \Phi\left(\phi \cdot d_2\right)\right], \quad d_{1,2} = \frac{\ln\left(F/K\right)}{\nu} \pm \frac{\nu}{2}$$
   and Φ being the cumulated standard normal distribution function.

   (b) Use put-call parity to derive the put option price
   $$\mathbb{E}\left[[K - S(T)]^+ \mid \mathcal{F}_t\right] = \text{Black}\left(S(t), K, \sigma\sqrt{T-t}, -1\right)$$

2. **Shifted Log-normal model**
   Consider the shifted log-normal model
   $$dS(t) = \frac{\sigma}{\lambda} \cdot \left[S(t) + \lambda\right] \cdot dW(t)$$
   with shift λ and re-scaled shifted log-normal volatility σ/λ.

   (a) Show that the shifted log-normal model converges to the normal model with normal volatility σ in the sense that
   $$\lim_{\lambda \to \infty} \mathbb{E}\left[[S(T) - K]^+ \mid \mathcal{F}_t\right] = \text{Bachelier}\left(S(t), K, \sigma\sqrt{T-t}, 1\right).$$

   (b) Confirm your results numerically with Python and illustrate the result.

3. **SABR model volatility approximation**
   Consider the SABR model with implied normal volatility approximation
   $$\sigma_N\left(S(t), K, T\right) = \nu \cdot \frac{S(t) - K}{\hat{\chi}(\zeta)} \cdot \left[1 + I^1\left(S(t), K\right) \cdot T\right]$$
   with
   $$S_{av} = \frac{S(t) + K}{2}, \quad \zeta = \frac{\nu}{\alpha} \cdot \int_K^{S(t)} \frac{dx}{C(x)}, \quad \hat{\chi}(\zeta) = \ln\left(\frac{\sqrt{1 - 2\rho\zeta + \zeta^2} - \rho + \zeta}{1 - \rho}\right),$$

$$I^1(K) = \frac{2\gamma_2 - \gamma_1^2}{24}\alpha^2 C(S_{av})^2 + \frac{\rho\nu\alpha\gamma_1}{4}C(S_{av}) + \frac{2-3\rho^2}{24}\nu^2,$$

$$\gamma_1 = \frac{C'(S_{av})}{C(S_{av})}, \quad \gamma_2 = \frac{C''(S_{av})}{C(S_{av})}.$$

More concrete, we get with $C(S) = S^\beta$ and $\beta \in (0,1)$

$$\zeta = \frac{\nu}{\alpha} \cdot \frac{S(t)^{1-\beta} - K^{1-\beta}}{1-\beta}, \quad \gamma_1 = \frac{\beta}{S_{av}}, \quad \gamma_2 = \frac{\beta(\beta-1)}{S_{av}^2}$$

(a) Based on the formula for σ_N, derive the volatility approximation for the normal SABR model with $\beta = 0$ via

$$\lim_{\beta \to 0} \sigma_N (S(t), K, T).$$

(b) Based on the formula for σ_N, derive the volatility approximation for the CEV model with $\beta > 0$ and $\nu = 0$ via

$$\lim_{\nu \to 0} \sigma_N (S(t), K, T).$$

(c) Show that, for $\beta \to 0$ and $\nu \to 0$ above approximation converges to the normal volatility, i.e.

$$\lim_{\beta,\nu \to 0} \sigma_N (S(t), K, T) = \alpha.$$

(d) Confirm your results numerically, e.g. via Python and illustrate the results.

4. **Shifted SABR model**
Consider the SABR model normal volatility approximation for general local volatility functions $C(S)$ in Exercise 3.

(a) Derive the approximation for the shifted SABR model with local volatility function

$$C(S) = (S + \lambda)^\beta$$

(b) Recall the model-independent pricing result for shifted processes $\tilde{S}(t) = S(t) - \lambda$ and Vanilla option pricing formula $V(S(t), K)$,

$$\tilde{V}(\tilde{S}(t), K) = \mathbb{E}\left[\left(\tilde{S}(T) - K\right)^+ \mid \mathcal{F}_t\right] = V\left(\tilde{S}(t) + \lambda, K + \lambda\right)$$

Is the approximation from (a) consistent to this model-independent result? Does your conclusion change if you replace the arithmetic average $S_{av} = \frac{S(t)+K}{2}$ by the geometric average $S_{av} = \sqrt{S(t) \cdot K}$?

(c) Modify the Python code in `src/sabr_model.py` to allow for a shift parameter λ. Analyse the approximation accuracy of the normal volatility formula from (a) compared to the model-implied normal volatilities obtained via Monte-Carlo simulation. Does approximation accuracy change if we incorporate a shift in the formula?