

Interest Rate Modelling and Derivative Pricing, SS 2023

Exercise 4

1. Affine term-structure models and HJM model representation

The continuously-compounded spot rate is defined as

$$R(t, T) = -\frac{\ln [P(t, T)]}{T - t}.$$

Affine term structure models are characterised by the fact that continuously-compounded spot rates take the form

$$R(t, T) = \alpha(t, T) + \beta(t, T) \cdot r(t)$$

for deterministic functions α and β .

- (a) Show that for affine models future zero bond prices can be represented as

$$P(t, T) = A(t, T) \cdot e^{-B(t, T) \cdot r(t)}$$

with deterministic functions A and B .

- (b) Assume a one-factor diffusion for the short rate in the risk-neutral measure via

$$dr(t) = [\theta(t) - \chi \cdot r(t)] \cdot dt + \sigma(t, r(t)) \cdot dW(t).$$

Show that the dynamics of the forward rates $f(t, T)$ become

$$df(t, T) = [\dots] \cdot dt + \frac{\partial B(t, T)}{\partial T} \cdot \sigma(t, r(t)) \cdot dW(t)$$

- (c) Now consider the case $\sigma(t, r(t)) = \sqrt{a + b \cdot r(t)}$. What are the dynamics of the model in terms of the state variables $x(t)$ and $y(t)$?

2. Swap rate dynamics in separable HJM model

Consider the one-factor separable HJM model with $r(t) = f(0, t) + x(t)$ and risk-neutral dynamics

$$\begin{aligned} dx(t) &= [y(t) - \chi \cdot x(t)] dt + \sigma_r(t) dW(t), \\ dy(t) &= [\sigma_r(t)^2 - 2\chi \cdot y(t)] dt. \end{aligned}$$

The forward swap rate (in the single curve setting) is (for $t \leq T_0$)

$$S(t) = \frac{P(t, T_0) - P(t, T_N)}{\sum_{i=1}^N \tau_i P(t, T_i)}.$$

Recall that zero bonds can be written in terms of state variables via $P(t, T) = P(x(t), y(t); t, T)$.

- (a) What are the dynamics of the swap rate in the annuity measure given that HJM model?
 (b) Assume the short rate volatility $\sigma_r(t)$ is a deterministic function. Can the swap rate best be approximated by a normal, lognormal or shifted log-normal model? What is the corresponding volatility parameters in such an approximated model?