Interest Rate Modelling and Derivative Pricing

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1 Sumary

In this lecture we discuss the modelling of interest rates and the pricing of interest rate derivatives. The guiding example will be the pricing and risk management of Bermudan swaptions, one of the most actively traded exotic interest rate derivatives in the market. Along the way we will present the various building blocks which are relevant for derivative pricing in general.

The lecture is structured in three blocks. In the first block we focus on the modelling of interest rates. We start by introducing quantities for static yield curve modelling and pricing of linear products. Then we derive the basic pricing models for Vanilla interest rate options (caps and European swaptions). This is supplemented by an analysis of the classical SABR model. In a next step we analyse the basic principles of term structure models in the Heath-Jarrow-Morton framework. This allows us to specialise to the classical Hull White interest rate model which will be discussed in detail. We also present relevant pricing methods applicable for Bermudan swaptions.

The second block is dedicated to the calibration of the interest rate models. We discuss the resulting optimization problems for static yield curve construction and Hull White model calibration. Moreover, we eleborate on the numerical sulution methods for the optimization problems in question. This discussion includes various aspects for improving computational efficiency and robustness of the algorithms relevant in practice.

In the third block we analyse sensitivity calculation. This is particularly important in practice since Delta and Vega sensitivities are the building blocks for hedging and risk management. We discuss the pro's and con's of the still widely-used finite difference approximation methods. Then we present the basic methodologies from Algorithmic Differentiation which become more and more applied in the financial industry.

The lecture follows in most parts the presentations in the three volumes of [1]. A further important reference is [2]. This will be complemented by references for special topics, e.g. [3, 4, 5, 6, 7].

References

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2 Draft Contents

2.1 Interest Rate Modelling

1. Recap and preliminaries ([1, Sec. 1.1 - 1.8], [2, Sec. 2])
   - Basic notation, relevant math finance results
   - Risk-neutral pricing, numeraire, change of meassure
2. Static yield curve modelling ([2, Sec. 1],[8])
   - Single-curve setting (discount factors, zero rates forward rates, year fractions, conventions)
   - Classical yield curve construction from deposits and swap rates
   - Tenor basis, interst rate indeces, FRAs and Vanilla swaps
   - Market forward Libor and swap rates
   - Modern multi-curve yield curve construction
3. Vanilla modells ([9])
   - Bachelier, Black, Shifted-Black model
   - Implied volatility smiles, volatility dynamics (volatility backbone)
   - SABR model
     - Specification and properties
     - Pricing methods (PDE, MC)
     - Asymptotic expansion for Vanilla options, arbitrage for negative strikes
   - (brief) SABR extensions for negative interest rates (Shifted SABR, ...)
4. HJM term structure modelling framework ([1, Sec. 4.4], [2, Sec. 5])
   - Bond price and forward rate dynamics
   - Gaussian HJM modells
   - Gaussian HJM models with Markovian short rate
5. Hull White modell ([2, Sec. 3.3])
   - dynamics, model properties (yield curve fit, future yield curves, mean reversion)

- analytic formulas for bonds and bond options
- incorporating tenor basis (simple versus continuous compounded spreads)
- equivalence swaption and bond options

6. Pricing methods for Bermudan swaptions ([2, Sec. 3.3.3], [10, Sec. 25], [3, Sec. 3.3, 8.6.1], [1, 18.3])
   - Bermudan swaption as max-European swaption plus switch option
   - Trinomial tree method, PDE (finite difference method, boundary conditions), Density integration
   - Monte Carlo simulation for short rate models
     - state diffusion, SDE integration
     - numeraire simulation, simulation in forward meassure
     - American Monte Carlo via regression

2.2 Model Calibration

1. Optimisation problem for yield curve calibration ([1, Sec. 6])
   - Curve parametrization and regularisation
   - Bootstrapping versus full optimisation
   - Multi-curve calibration and dependencies
2. Optimisation problem for Hull White Model ([1, Sec. 10.1.4])
   - Short rate volatility calibration
   - Mean reversion calibration (auto-correlation, swaption volatility ratio, switch value)
3. Numerical methods for calibration ([4], [5])
   - Newton's method, Levenberg-Marquardt
   - Jacobian-free methods
     - 1-dim. secant method, Brent's algorithm
     - Rank-1 updates (Broyden, Adjoint Broyden)
   - Stabilisation methods for optimization algorithms
     - parameter transformation
     - line search methods
     - Tikhonov regularization

2.3 Sensitivity Calculation

1. Yield curve deltas ([1, Sec. 6.4])
   - Benchmark-rate versus zero/forward rate sensitivities
   - Jacobian method for risk transformation
2. Vega and smile sensitivities
   - Swaption Vega versus short rate Vega
   - Risk aggregation

- Vega contribution to Delta (smile and volatility backbone)

3. Numerical methods for sensitivitiy calculation ([6], [7], [1, Sec. 24]

    - Finite difference approximation via bump-and-reval (convergence versus stability)
    - Sensitivities via Algorithmic Differentiation
        - basic principles of AD (chain rule applied to elementary operations)
        - forward (tangent) mode versus reverse (adjoint) mode; use cases and complexity
    - Path-wise sensitivities for Monte-Carlo methods