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A Law of Large Numbers for Limit Order Books

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We define a stochastic model of a two-sided limit order book in terms of its key quantities *best bid [ask] price* and the *standing buy [sell] volume density*. For a simple scaling of the discreteness parameters, that keeps the expected volume rate over the considered price interval invariant, we prove a limit theorem. The limit theorem states that, given regularity conditions on the random order flow, the key quantities converge in probability to a tractable continuous limiting model. In the limit model the buy and sell volume densities are given as the unique solution to first-order linear hyperbolic PDEs, specified by the expected order flow parameters. We calibrate order flow dynamics to market data for selected stocks and show how our model can be used to derive endogenous shape functions for models of optimal portfolio liquidation under market impact.

Key words: limit order book, scaling limit, averaging principle, queueing theory
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History:

Introduction A significant part of financial transactions is nowadays carried out through electronic limit order books (LOBs). A LOB is a record, maintained by an exchange or specialist, of unexecuted orders awaiting execution. From a mathematical perspective, LOBs are high-dimensional complex priority queueing systems. Incoming limit orders can be placed at many different price levels while incoming market orders are matched against standing limit orders according to a set of priority rules. Almost all exchanges give priority to orders submitted at more competitive prices ("price priority") and displayed orders have priority over hidden orders at the same level ("display priority"). Orders with the same display status and submission price are usually served on a first-come-first-serve basis. The inherent complexity of limit order books renders their mathematical analysis challenging. In this paper, we propose a queueing theoretic LOB model whose dynamics converges to a coupled ODE:PDE system after suitable scaling that can be solved in closed form.

There is a significant economic and econometric literature on LOBs including Biais at al. [7], Easley and O'Hara [13], Foucault et al. [15], Gloston and Milgrom [18], Parlour [25], Rosu [27] and many others that puts a lot of emphasis on the realistic modeling of the working of the LOB, and on its interaction with traders' order submission strategies. There are, however, only few papers that analyze order flows and the resulting LOB dynamics in a mathematically rigorous manner. Among the first was the one by Kruk [22]. He studied a queueing theoretic model of a transparent double auction in continuous time. The microstructure may be interpreted as that of a simple limit order book, if one considers a buyer as a buy limit order and a seller as a sell limit order. At the auction, there are $N \in \mathbb{N}$ possible prices of the security and thus 2N different classes of customers (N classes of buyers and N classes of sellers). Kruk established diffusion and fluid limits. The diffusion limit states that for N=2 possible prices, the scaled number of outstanding buy orders at the lower price and the scaled number of outstanding sell orders at the higher price converge weakly to a semimartingale reflected two-dimensional Brownian motion in the first quadrant. The fluid limit is such that various LOB quantities converge weakly to affine functions of time.

Since information on the best bid and ask price and volume at different price levels is available to all market participants, it is natural to assume that the order arrival dynamics depends on the current state of the order book. The feature of conditional state-dependence was considered by Cont et al. [12], who proposed a continuous time stochastic model with a finite number of possible prices where events (buy/sell market order arrival, buy/sell limit order placement and cancelation) are modeled using independent Poisson processes. The arrival rates of limit orders depend on the distance to the best bid and ask price in a power-law fashion. The authors were able to show that the state of the order book, defined as a vector containing all volumes in the order book at different prices, is an ergodic Markov process. Using this fact, several key quantities such as the probabilities of a mid price move, a move in the bid price before a move in the ask price, or the probability of volume execution before a price move could be computed and benchmarked against real data without taking scaling limits.

Cont and de Larrard [11] considered a scaling limit in the diffusion sense for a Markovian limit order market in which the state is represented by the best bid and ask price and the queue length, i.e. the number of orders at the best bid and ask price, respectively. With this reduction of the state space, under symmetry conditions on the spread and stationarity assumptions on the queue lengths, it was shown that the price converges to a Brownian motion with volatility specified by the model parameters in the diffusion limit. Very recently, Cont and de Larrard [10] studied the reduced state space under weaker conditions and proved a refined diffusion limit by showing that under heavy traffic conditions the bid and ask queue lengths are given by a two-dimensional Brownian motion in the first quadrant with reflection to the interior at the boundaries, similar to the diffusion limit result for $N = 2$ prices in [22].

In the framework analyzed by Abergel and Jedidi [1], the volumes of the order book at different distances to the best bid and ask were modeled as a finite dimensional continuous time Markov chain and the order flow as independent Poisson processes. Under the assumption that the width of the spread is constant in time, using Foster-Lyapunov stability criteria for the Markov chain, the authors proved ergodicity of the order book and a diffusion limit for the mid price. In the diffusion limit, the mid price is a Brownian motion with constant volatility given by the averaged price impact of the model events on the order book.

In this paper, we prove a law of large numbers result for the whole book (prices and volumes). Specifically, we propose a continuous-time model of a two-sided state-dependent order book with random order flow and cancelation, and countably many submission price levels. The buy and the sell side volumes are coupled through the best bid and ask price dynamics.[1] We model the buy and sell side volumes as *density functions in relative price coordinates*, i.e. relative to their distance to the best bid/ask prices. Volumes at positive distances are limit orders awaiting execution ("visible book"); volumes at negative distances specify orders that *would* be placed in the spread if the next

[1] The coupling of the buy and sell sides through prices is essential for the limiting volume dynamics to follow a PDE. It is *not* essential for obtaining a scaling limit per se. Our mathematical framework is flexible enough to allow for dependencies of order flows on standing volumes, but that one would lead to a function-valued ODE as the scaling limit, rather than a PDE. The PDE-scaling is so much more transparent that it justifies, in our view, the restriction of order flow dependencies on prices.

order is a spread limit order placement ("shadow book"). The state of the book at any point in time is thus described by a quadruple comprising the best bid price, the best ask price, the relative buy volume density function and the relative sell volume density function.

The state dynamics is defined in terms of a recursive stochastic process taking values in a function space. When the analysis of the market is limited to prices as in e.g. Garman [17], Bayraktar et al. [6] or Horst and Rothe [21], or to the joint dynamics of prices and *aggregate volumes* (e.g. at the top of the book) as in Cont and de Larrard [10, 11], then the limiting dynamics can naturally be described by ordinary differential equations or real-valued diffusion processes, depending on the choice of scaling. The analysis of the whole book including the *distribution* of standing volume across many price levels is much more complex. Osterrieder [24] modeled LOBs using measure-valued diffusions. Our approach is based on an averaging principle for Banach space-valued processes. The key is a uniform law of large numbers for Banach space-valued triangular martingale difference arrays. It allows us to show that the volume densities take values in L^2 and that the noise in the order book models vanishes in the limit with our choice of scaling.

Our scaling limit requires two time scales: a fast time scale for cancelations and limit order placements outside the spread (events that do not lead to price changes), and a comparably slow time scale for market order arrivals and limit order placements in the spread (events that lead to price changes). The choice of time scales captures the fact that in real-world markets significant proportions of orders are never executed. Mathematically, the different time scales imply that aggregate cancelations and limit order placements outside the spread in between two consecutive price changes can be approximated by their expected values.

Our main result states that when the price tick tends to zero, order arrival rates tend infinity, and the impact of an individual placement/cancelation on the standing volume tends to zero, then the sequence of scaled order book dynamics converges in probability uniformly over compact time intervals to a deterministic limit. The limiting model is such that the best bid and ask price dynamics can be described in terms of two coupled ODEs, while the dynamics of the relative buy and sell volume density functions can be described in terms of two linear first-order hyperbolic PDEs with variable coefficients. A similar limiting dynamics was recently obtained in [16] but with constant limiting price process. The latter is not required in our model.

Our LOB model can be used to obtain shape functions for models of optimal portfolio liquidation under market impact. In such models the goal is to find optimal strategies for unwinding large numbers of shares over small time periods. They typically assume that the dynamics of standing buy (or sell) side volumes can be described in terms of exogenous shape functions. Within our modeling framework shape function arise as part of the solution to a coupled ODE:PDE system. Calibrating the model parameters to market data, this allows for a fully endogenous derivation of shape functions from order arrival and cancellation dynamics. Using LOBSTER data data for Jan 2, 2014 we calibrated order placement dynamics for selected stocks. For the stocks Ebay and Facebook we found that exponential densities provide good fits; for Apple the placement densities turn out to be almost constant. For Microsoft most of the placement and cancellation activity concentrate around the top of the book. For such stocks, our scaling does not seem to be appropriate. We further report empirical evidence supporting our assumption of multiple time scales. For instance, for Apple we found that only 1.6% of all orders on Jan 2, 2014 lead to price changes.

The remainder of this paper is organized as follows. In Section 1 we define a sequence of limit order book models in terms of four scaling parameters: price tick, expected waiting time between two consecutive orders, volume placed/canceled and the proportion of order arrivals leading to price changes, and state our main result. In Section 2 we establish convergence of the bid/ask price dynamics to a 2-dimensional ODE. Section 3 is devoted to the analysis of the limiting volume dynamics. Section 4 illustrates how our model can potentially be applied in a portfolio liquidation framework; Section 5 illustrates how selected model parameters can be calibrated to market data.

The uniform law of large numbers for triangular martingale difference arrays as well as useful auxiliary results are proved in an appendix.

1. A sequence of discrete order book models In electronic markets orders can be submitted for prices that are multiples of the *price tick*, the smallest increment by which the price can move. In this section, we introduce a sequence of order book models for which we establish a scaling limit when the price tick and impact of a single order on the state of the book tend to zero, while the rate of order arrivals tends to infinity.

The sequence of models is indexed by $n \in \mathbb{N}$. We assume that the set of price levels at which orders can be submitted in the n:th model is $\{x_j^{(n)}\}_{j\in\mathbb{Z}}$ where $\mathbb{Z}$ denotes the one-dimensional integer lattice.[2] We put $x_j^{(n)} := j \cdot \Delta x^{(n)}$ for $j \in \mathbb{Z}$ where $\Delta x^{(n)}$ is the tick size in the n:th model.

The *state* of the book changes due to incoming orders and cancelations. The state after $k \in \mathbb{N}$ such *events* will be described by a random variable $S_k^{(n)}$ taking values in a suitable *state space* E. In the n:th model, the k:th event occurs at a random point in time $\tau_k^{(n)}$. The time between two consecutive events will be tending to zero sufficiently fast as $n \to \infty$. The state and time dynamics will be defined, respectively, as

$$S_0^{(n)} := s_0^{(n)}, \quad S_{k+1}^{(n)} := S_k^{(n)} + \mathcal{D}_k^{(n)}(S_k^{(n)}) \tag{1}$$

and

$$\tau_0^{(n)} := 0, \quad \tau_{k+1}^{(n)} := \tau_k^{(n)} + \mathcal{C}_k^{(n)}(S_k^{(n)}). \tag{2}$$

Here $s_0^{(n)} \in E$ is a deterministic initial state, and $\mathcal{D}_k^{(n)}(S_k^{(n)}) : E \to E$ and $\mathcal{C}_k^{(n)}(S_k^{(n)}) : E \to [0,\infty)$ are random operators that will be introduced below. The conditional expected increment of the state sequence, given $S_k^{(n)}$, will be denoted $\mathbb{E}\left[\mathcal{D}_k^{(n)}(S_k^{(n)})\right]$; the unconditional increment $\mathbb{E}\left[\mathcal{D}_k^{(n)}\right]$.

1.1. The order book models In the sequel we specify the dynamics of our order book models. Throughout, all random variables will be defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

1.1.1. The initial state The initial state of the book in the n:th model is given by a pair $(B_0^{(n)}, A_0^{(n)})$ of best bid and ask prices together with the standing buy and sell limit order volumes at the various price levels. It will be convenient to identify the standing volumes with step functions

$$v_{b,0}^{(n)}(x) := \sum_{j=0}^{\infty} v_{b,0}^{(n),j} \mathbf{1}_{[x_j^{(n)}, x_{j+1}^{(n)})}(x), \quad v_{s,0}^{(n)}(x) := \sum_{j=0}^{\infty} v_{s,0}^{(n),j} \mathbf{1}_{[x_j^{(n)}, x_{j+1}^{(n)})}(x) \quad (x \geq 0)$$

that specify the liquidity available for buying and selling *relative* to the best bid and ask price. The liquidity available for buying (sell side of the book) $j \in \mathbb{N}_0$ ticks *above* the best ask price at the price level $x^{(n)}_{A_0^{(n)}+j}$ is

$$\int_{x_j^{(n)}}^{x_{j+1}^{(n)}} v_{s,0}^{(n)}(x)\mathrm{d}x = v_{s,0}^{(n),j} \cdot \Delta x^{(n)}.$$

The volume available for selling (buy side of the book[3]) $l \in \mathbb{N}_0$ ticks *below* the best bid price at the price level $x^{(n)}_{B_0^{(n)}-l}$ is

$$\int_{x_l^{(n)}}^{x_{l+1}^{(n)}} v_{b,0}^{(n)}(x)\mathrm{d}x = v_{b,0}^{(n),l} \cdot \Delta x^{(n)}.$$

In order to conveniently model placements of limit orders into the spread, we extend $v_{b,0}^{(n)}$ and $v_{s,0}^{(n)}$

[2] The assumption that there is no minimum price is made for analytical convenience and can easily be relaxed.

[3] Notice that the liquidity available for buying is captured by the sell side of the book and vice versa.

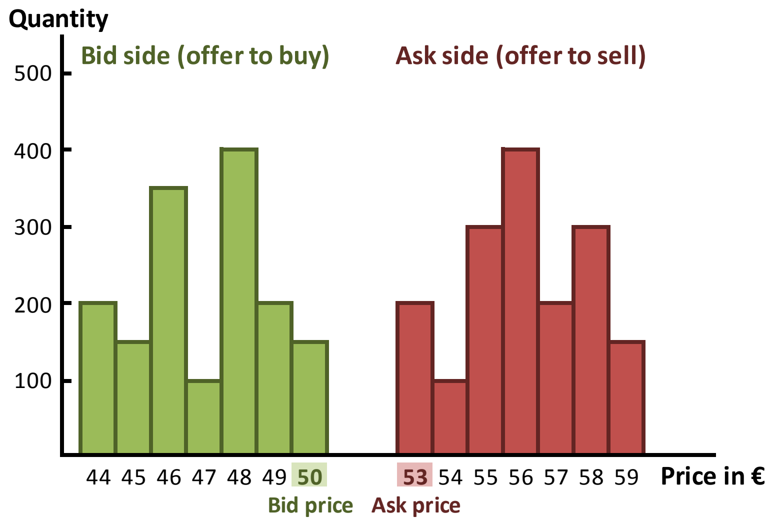

FIGURE 1. Snapshot of the order book.

to the negative half-line. The collection of volumes standing at negative distances from the best bid/ask price is referred to as the *shadow book*. The shadow book will undergo the same dynamics as the standing volume ("visible book"). At any point in time it specifies the volumes that will be placed into the spread should such an event occur next[4]. The role of the shadow will be further illustrated in Section 1.1.4 below; see also Figure 3.

DEFINITION 1. In the n:th model the initial state of the book is given by a quadruple

$$S_0^{(n)}(\cdot) = \left(B_0^{(n)}, A_0^{(n)}, v_{b,0}^{(n)}(\cdot), v_{s,0}^{(n)}(\cdot)\right)$$

where $B_0^{(n)} \leq A_0^{(n)}$ are the best bid/ask price and the step functions $v_{b,0}^{(n)}, v_{s,0}^{(n)} : \mathbb{R} \to [0,\infty)$ are to be interpreted as follows:

$$v_{b,0}^{(n)}(x)\ \left[v_{s,0}^{(n)}(x)\right] := \begin{cases} \text{standing buy [sell] limit order volume density at price distance } x \\ \text{below [above] the best bid [ask] price, for } x \geq 0 \text{ (visible book)} \\ \\ \text{potential buy [sell] limit order volume density at price distance } x \\ \text{above [below] the best bid [ask] price, for } x < 0 \text{ (shadow book).} \end{cases} \tag{3}$$

Throughout, we shall use the notation $f = \mathcal{O}(g)$ and $f = o(g)$ to indicate that the function f grows asymptotically no faster than g, respectively that $|f(x)/g(x)| \to 0$ as $x \to \infty$. With this, we are ready to state our conditions on the initial states. In particular, we assume that the initial volume density functions vanish outside a compact price interval.[5]

[4] One has to specify the volumes placed into the spread somehow. Our choice of shadow books is one such way. The role of the shadow book will be further clarified in the following subsection when we define the impact of order arrivals on the state of the book. Its initial state is part of the model; future states will undergo the dynamics analogous to those of the visible book.

[5] This assumption, which may be generalized, considerably simplifies some of the analysis that follows.

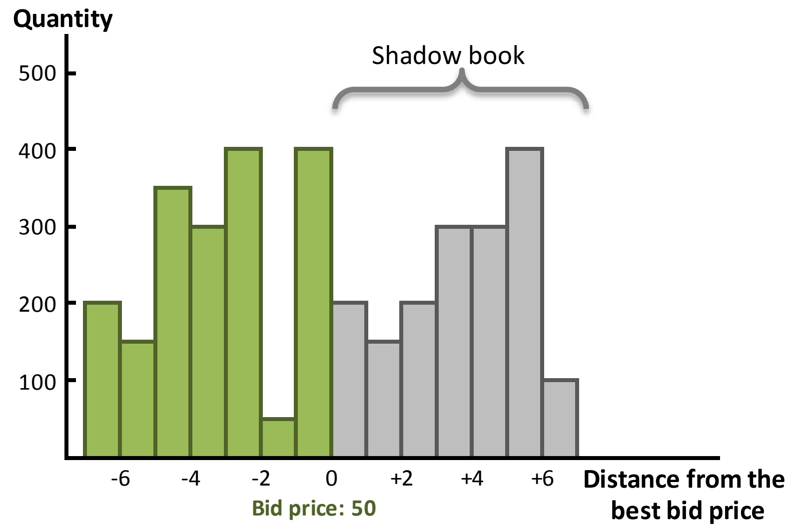

FIGURE 2. Buy-side volume density function in relative coordinates; green: standing ("visible") volume; grey: shadow book.

ASSUMPTION 1 **(Convergence of initial states)**. *The initial volume density functions vanish outside a compact interval $[-M, M]$, for some $M > 0$. Moreover, there exists non-negative bounded and continuously differentiable functions $v_{r,0} \in L^2$ $(r \in \{b, s\})$ with bounded derivatives such that*

$$||v_{r,0}^{(n)} - v_{r,0}||_{L^2} = o(1)$$

as well as

$$|v_{r,0}^{(n)}(\cdot \pm \Delta x^{(n)}) - v_{r,0}^{(n)}(\cdot)|_\infty = \mathcal{O}(\Delta x^{(n)}).$$

Here, $\|\cdot\|_{L^2}$ denotes the L^2-norm on $\mathbb{R}$ with respect to Lebesgue measure. Moreover,

$$\lim_{n\to\infty}(B_0^{(n)}, A_0^{(n)}) = (B, A).$$

The first condition on the volume density functions is intuitive. The second condition will become clear later; it will be used to bound the impact of market orders and limit orders placed into the spread on the state of the book.

1.1.2. Event types There are eight events - labeled **A, ..., H** - that change the state of the book. The events **A, ..., D** affect the buy side of the book:

$$\mathbf{A} := \{\text{market sell order}\} \qquad \mathbf{B} := \{\text{buy limit order placed in the spread}\}$$
$$\mathbf{C} := \{\text{cancelation of buy volume}\} \qquad \mathbf{D} := \{\text{buy limit order not placed in spread}\}$$

The remaining four events affect the sell side of the book:

$$\mathbf{E} := \{\text{market buy order}\} \qquad \mathbf{F} := \{\text{sell limit order placed in the spread}\}$$
$$\mathbf{G} := \{\text{cancelation of sell volume}\} \qquad \mathbf{H} := \{\text{sell limit order not placed in the spread}\}.$$

We will describe the state dynamics of the n:th model by a stochastic process $\{S_k^{(n)}\}_{k\in\mathbb{N}}$ that takes values in the Hilbert space

$$E := \mathbb{R} \times \mathbb{R} \times L^2 \times L^2.$$

The first two components of the vector $S_k^{(n)}$ stand for the best bid and ask price after k events; the third and fourth component refer to the buy and sell volume density functions relative to the best bid and ask price, respectively (visible and shadow book). We define a norm on E by

$$\|\alpha\|_E := |\alpha_1| + |\alpha_2| + \|\alpha_3\|_{L^2} + \|\alpha_4\|_{L^2}, \quad \alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in E. \tag{4}$$

In the sequel we specify how different events change the state of the book and how order arrival times and sizes scale with the parameter $n \in \mathbb{N}$.

1.1.3. Active orders Market orders and placements of limit orders in the spread lead to price changes.[6] With a slight abuse of terminology we refer to these order types as *active orders*. For convenience, we assume that market orders match precisely against the standing volume at the best prices and that limit orders placed in the spread improve prices by one tick. The assumption that market orders decrease (increase) the best bid (ask) price by one tick while limit orders placed in the spread decrease (increase) prices by the same amount has been made in the literature before and can be generalized without too much effort. However, this would unnecessarily complicate the analysis that follows.

Remark 1. A market order whose size exceeds the standing volume at the top of the book and that would hence move the price by more than one tick is split by the exchange into a series of consecutively executed orders. The size of each such 'suborder', except the last, equals the liquidity at the current best price. Thus, by definition, a single market order cannot move the price by more than one tick.

If the k:th event is a sell market order (Event **A**), then the relative buy volume density shifts one price tick to the left (the liquidity that stood l ticks into the book now stands $l-1$ ticks into the book), the best bid price decreases by one tick and the relative sell volume density and the best ask price remain unchanged. Since the relative volume density functions are defined on the whole real line, the transition operators

$$T_+^{(n)}(v)(\cdot) = v(\cdot + \Delta x^{(n)}), \quad T_-^{(n)}(v)(\cdot) = v(\cdot - \Delta x^{(n)})$$

are well defined and one has that

$$v_{b,k+1}^{(n)}(\cdot) = T_+^{(n)}\left(v_{b,k}^{(n)}\right)(\cdot), \quad v_{s,k+1}^{(n)}(\cdot) = v_{s,k}^{(n)}(\cdot)$$

and

$$B_{k+1}^{(n)} = B_k^{(n)} - \Delta x^{(n)}, \quad A_{k+1}^{(n)} = A_k^{(n)}.$$

The placement of orders into the spread will be modeled using the shadow book. If the k:th event is a buy limit order placement in the spread (Event **B**), the relative buy volume density shifts one price tick to the right (the liquidity that stood one tick above the best bid in the shadow book now stands at the top of the visible book), the best bid price increases by one tick and the relative buy volume density and the best ask price remain unchanged:

$$v_{b,k+1}^{(n)}(\cdot) = T_-^{(n)}\left(v_{b,k}^{(n)}\right)(\cdot), \quad v_{s,k+1}^{(n)}(\cdot) = v_{s,k}^{(n)}(\cdot)$$

and

$$B_{k+1}^{(n)} = B_k^{(n)} + \Delta x^{(n)}, \quad A_{k+1}^{(n)} = A_k^{(n)}.$$

Remark 2. Notice that market order arrivals and limit order placements in the spread are "inverse operations": a market sell order arrival followed by a limit buy order placement in the spread (or vice versa) leaves the book unchanged.

[6] A market order that does not lead to a price change can be viewed as a cancelation of standing volume at the best bid/ask price.

1.1.4. Passive orders Limit order placements outside the spread and cancelations of standing volume do not change prices. With the same minor abuse of terminology as before, we refer to these order types as *passive orders*. We assume that cancelations of buy volume (Event **C**) occur for random *proportions* of the standing volume at random price levels while limit buy order placements outside the spread (Event **D**) occur for random *volumes* at random price levels. The submission and cancelation price levels are chosen *r*elative to the best bid price.

Assumption 2. *For each $k \in \mathbb{N}$ there exist random variables ω_k^C, ω_k^D taking values in $(0,1)$, respectively $[0,M]$ for some $M>0$ and random variables π_k^C, π_k^D taking values in $[-M,M]$ such that, if the k:th event is a limit buy order cancelation/placement, then it occurs at the price level $x^{(n)}_{B_k^{(n)}-j}$ $(j \in \mathbb{Z})$ for which*

$$\pi_k^{C,D} \in [x_j^{(n)}, x_{j+1}^{(n)}).$$

The volume canceled, respectively, placed is

$$\omega_k^C \cdot \Delta v^{(n)} \cdot v_{b,k}^{(n)}(\pi_k^C) \quad \textit{respectively} \quad \omega_k^D \cdot \Delta v^{(n)}.$$

Here $v_{b,k}^{(n)}(\pi_k^C)$ is the value of the volume density function at the cancelation price level, and $\Delta v^{(n)}$ is a scaling parameter that describes the impact of an incoming limit order (cancelation) on the state of the order book.

Volume changes take place in the visible or the shadow book, depending on the sign of of π_k^I. If $\pi_k^I \geq 0$, then the visible book changes; if $\pi_k^I < 0$, then the placement/cancelation takes place in the shadow book. In order to illustrate the working of the shadow book, suppose that $(k+1)$-st event is an order placement of size $\omega_{k+1}^D \Delta v^{(n)}$ in the (buy-side) shadow book one tick above the best bid, i.e. $\pi_{k+1}^D \in [-\Delta x^{(n)}, 0)$ and that the $(k+2)$nd event is a buy limit order placement in the spread. Then,

$$B_{k+2}^{(n)} = B_{k+1}^{(n)} + \Delta x^{(n)} = B_k^{(n)} + \Delta x^{(n)}$$

and the value of the volume density function at the top of the book is:

$$v_{b,k+2}^{(n),0} = v_{b,k+1}^{(n),-1} = v_{b,k}^{(n),-1} + \omega_{k+1}^D \frac{\Delta v^{(n)}}{\Delta x^{(n)}}.$$

This is how the buy order previously placed into the shadow is now part of the visible book. The role of the shadow book is further illustrated by Figure 3. When a cancelation occurs at this price

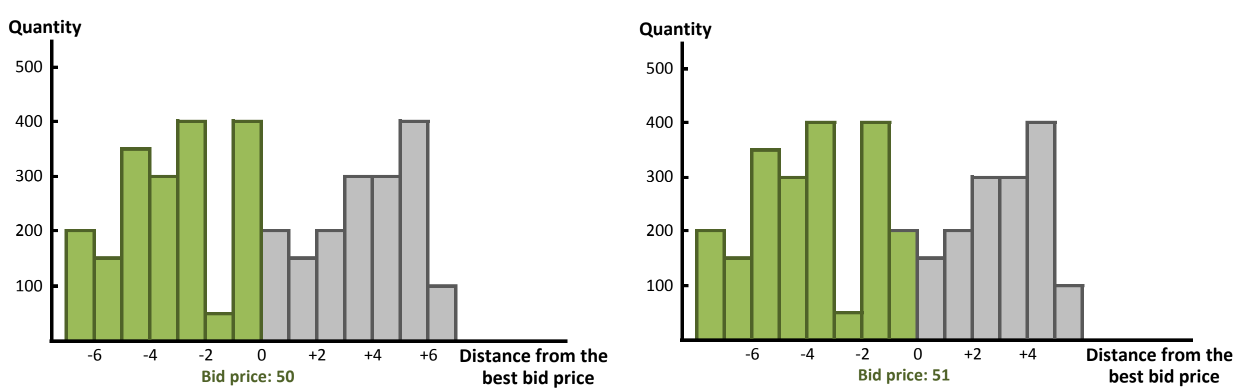

Figure 3. Left: initial buy side volume density with shadow book; right: new state (visible and shadow book) after a limit buy-order has been placed in the spread.

level, then the new volume is $(\Delta x^{(n)} - \omega_k^C \cdot \Delta v^{(n)}) \cdot v_{b,k}^{(n)}(x_j^n) = (1 - \omega_k^C \frac{\Delta v^{(n)}}{\Delta x^{(n)}}) \cdot v_{b,k}^{(n)}(x_j^n) \cdot \Delta x^{(n)}$ because cancellations are proportional to standing volumes. On the level of the volume density functions this yields:

$$v_{b,k+1}^{(n)}(\cdot) = v_{b,k}^{(n)}(\cdot) - \frac{\Delta v^{(n)}}{\Delta x^{(n)}} \cdot M_k^{(n),C}(\cdot) \cdot v_{b,k}^{(n)}(\cdot), \quad \text{where} \quad M_k^{(n),C}(x) = \omega_k^C \sum_{j=-\infty}^{\infty} \mathbf{1}_{\{\pi_k^C \in [x_j^{(n)} x_{j+1}^{(n)})\}}(x).$$

Volume placements are additive. If the next order is a limit buy order, then the volume density function changes according to

$$v_{b,k+1}^{(n)}(\cdot) = v_{b,k}^{(n)}(\cdot) + \frac{\Delta v^{(n)}}{\Delta x^{(n)}} \cdot M_k^{(n),D}(\cdot), \quad \text{where} \quad M_k^{(n),D}(x) := \omega_k^D \sum_{j=-\infty}^{\infty} \mathbf{1}_{\{\pi_k^D \in [x_j^{(n)}, x_{j+1}^{(n)})\}}(x).$$

In either case, the bid/ask price and standing sell side volume of the book remain unchanged:

$$v_{s,k+1}^{(n)}(\cdot) = v_{s,k}^{(n)}(\cdot), \quad B_{k+1}^{(n)} = B_k^{(n)}, \quad A_{k+1}^{(n)} = A_k^{(n)}.$$

Similar considerations apply to the sell side with respective random quantities ω_k^G, ω_k^H and π_k^G, π_k^H.

Assumption 3. *For $I \in \{\mathbf{C}, \mathbf{D}, \mathbf{G}, \mathbf{H}\}$ the sequences $\{\omega_k^I\}_{k \in \mathbb{N}}$ and $\{\pi_k^I\}_{k\in\mathbb{N}}$ are independent sequences of i.i.d. random variables. Moreover, the random variables π_k^I have C^2-densities f^I with compact support.*

Lipschitz continuity of f^I implies existence of a constant $K < \infty$ such that:

$$\begin{aligned}\left|\mathbb{P}[\pi_k^I \in [x_j^{(n)}, x_{j+1}^{(n)})] - \mathbb{P}[\pi_k^I \in [x_{j-1}^{(n)}, x_j^{(n)})]\right| &\le \int_{x_{j-1}^{(n)}}^{x_j^{(n)}} |f^I(y + \Delta x^{(n)}) - f^I(y)| dy \\ &\le K \left(\Delta x^{(n)}\right)^2.\end{aligned} \tag{5}$$

Moreover, if we put

$$f^{(n),I}(\cdot) := \sum_{j=-\infty}^{\infty} f_j^{(n),I} \mathbf{1}_{[x_j^{(n)}, x_{j+1}^{(n)})}(\cdot) \quad \text{with} \quad f_j^{(n),I} := \frac{\mathbb{E}[\omega_1^I]}{\Delta x^{(n)}} \int_{x_j^{(n)}}^{x_{j+1}^{(n)}} f^I(x)\, \mathrm{d}x, \tag{6}$$

then

$$\|f^{n,I} - f^I\|_\infty = o(1) \quad \text{and} \quad \|T_{\pm}^{(n)} \circ f^{n,I} - f^{(n),I}\|_\infty = \mathcal{O}(\Delta x^{(n)}). \tag{7}$$

1.1.5. Event times The dynamics of event times is specified in terms of a sequence of inter-arrival times whose distributions may depend on the prevailing best bid and ask prices.

Assumption 4. *Let $\{\varphi_k\}_{k\in\mathbb{N}}$ be a sequence of non-negative random variables that are conditionally independent and identically distributed, given the current best bid and ask price:*

$$\mathbb{P}[\varphi_k \le t | S_k^{(n)}] = \mathbb{P}[\varphi_k \le t | B_k^{(n)}, A_k^{(n)}].$$

In the sequel we write $\varphi(A_k, B_k)$ for φ_k to indicate the dependence of the distribution of φ_k on the best bid and ask price. Similar notation will be applied to other random variables whenever convenient. In the n:th model, we scale time by a factor $\Delta t^{(n)}$. More precisely, we assume that the dynamics of the event times in the n:th model is given by:

$$\tau_{k+1}^{(n)} = \tau_k^{(n)} + \mathcal{C}_k^{(n)}(B_k^{(n)}, A_k^{(n)}), \quad \text{where} \quad \mathcal{C}_k^{(n)}(B_k^{(n)}, A_k^{(n)}) := \varphi(B_k^{(n)}, A_k^{(n)}) \cdot \Delta t^{(n)}. \tag{8}$$

1.1.6. Event types Event types are described in terms of a sequence of random *event indicator variables* $\{\phi_k\}_{k\in\mathbb{N}}$ taking values in the set $\{\mathbf{A},...,\mathbf{H}\}$. We assume that the random variables

$$\phi_k = \phi(B_k^{(n)}, A_k^{(n)}) \quad (k \in \mathbb{N}_0)$$

are conditionally independent and identically distributed, given the best bid/ask price, and that their conditional probabilities

$$p^{(n),I}(B_k^{(n)}, A_k^{(n)}) =: \mathbb{P}[\phi_k = I | S_k^{(n)}]$$

satisfy the following condition.

ASSUMPTION 5. *There exist continuous functions with bounded gradients $p^I : \mathbb{R}^2 \to [0,1]$ and a scaling parameter $\Delta p^{(n)} \to 0$ such that*

$$\begin{aligned}
p^{(n),I}(\cdot,\cdot) &= \Delta p^{(n)} \cdot p^I(\cdot,\cdot) \qquad \textit{for} \quad I = \mathbf{A},\mathbf{B},\mathbf{E},\mathbf{F} \\
p^{(n),I}(\cdot,\cdot) &= (1-\Delta p^{(n)}) \cdot p^I(\cdot,\cdot) \quad \textit{for} \quad I = \mathbf{C},\mathbf{D},\mathbf{G},\mathbf{H} \\
p^A + p^B + p^E + p^F &= 1 \\
p^C + p^D + p^G + p^H &= 1
\end{aligned}$$

REMARK 3. The preceding assumption implies that an event is an active order with probability $\Delta p^{(n)}$ and a passive order with probability $1 - \Delta p^{(n)}$, independently of the state of the book. Conditioned on the order being active or passive, it is of type I with a probability $p^I(\cdot,\cdot)$ that depends on the current best bid and ask price. We allow the above probabilities to be zero in order to account for the fact that no price improvements can take place when $B_k^{(n)} = A_k^{(n)}$ and to avoid depletion of the order book.[7]

The expected impact of each active order on the state of the book will be of the order $\Delta x^{(n)}$, and that of a passive order will be of the order $\Delta v^{(n)}$. Because active orders arrive at a rate that is $\Delta p^{(n)}$-times slower than that of passive orders, the relative average impact of active to passive orders on the state of the book will of the order $\frac{\Delta p^{(n)}\Delta x^{(n)}}{\Delta v^{(n)}}$. Our scaling limit requires to equilibrate the impact of active and passive orders. In order to guarantee that there will be no fluctuations in the standing volumes in the limit as $n \to \infty$ we also need a minimum relative frequency of passive order arrivals. This motivates the following assumption.

ASSUMPTION 6. *The scaling constants $\Delta p^{(n)}$, $\Delta x^{(n)}$, $\Delta v^{(n)}$ and $\Delta t^{(n)}$ are such that:*

$$\lim_{n\to\infty} \frac{\Delta x^{(n)} \cdot \Delta p^{(n)}}{\Delta v^{(n)}} = c_0, \quad \lim_{n\to\infty} \frac{\Delta v^{(n)}}{\Delta t^{(n)}} = c_1, \quad \textit{and} \quad \lim_{n\to\infty} \frac{\Delta p^{(n)}}{\left(\Delta t^{(n)}\right)^{\alpha}} = c_2$$

for some $\alpha \in \left(\frac{1}{2}, 1\right)$ and constants $c_0, c_1, c_2 > 0$.[8]

[7] For simplicity we assumed that the the initial volume density functions vanish outside a compact price interval. Hence there is a positive probability of depletion unless one assumes that no further buy/sell side price improvements take place if the distance of the current best bid/ask price from the initial state exceeds some threshold.

[8] For the results that follow, we will assume that $\frac{\Delta x^{(n)}\cdot\Delta p^{(n)}}{\Delta v^{(n)}} = 1$ and $\frac{\Delta v^{(n)}}{\Delta t^{(n)}} = 1$ as $n \to \infty$. Any other constant would require further constants in the limiting dynamics.

1.1.7. Active order times The previous two assumptions introduce two different time scales for order arrivals: a fast time scale for passive order arrivals, and a comparably slow time scale for active order arrivals. Inter-arrival times between passive orders are of the order $\Delta t^{(n)}$ while inter-arrival times between active orders are of the order $\Delta x^{(n)}$. In order to see this, let us denote by $\sigma_k^{(n)}$ the arrival time of the k:th active order. The *number* $r_{k+1}^{(n)}$ of events until the $(k+1)$-st active order arrival can be viewed as the first success time in a series of Bernoulli experiments with success probability $\Delta p^{(n)}$ and expected value $\frac{1}{\Delta p^{(n)}}$. The $(k+1)$-st active order arrives at time

$$\sigma_{k+1}^{(n)} = \sigma_k^{(n)} + \zeta_k^{(n)} \cdot \Delta x^{(n)}$$

where

$$\zeta_k^{(n)} := \sum_{l=\sigma_k^{(n)}+1}^{r_{k+1}^{(n)}-1} \varphi_l \cdot \Delta p^{(n)}.$$

Since the random variables $\varphi_{\sigma_k^{(n)}+1}, \ldots, \varphi_{r_{k+1}^{(n)}-1}$ are conditionally independent and identically distributed, $\{r_k^{(n)}\}$ and $\{\varphi_k\}$ are independent sequences. Because $\mathbb{E}[r_k^{(n)}] = \frac{1}{\Delta p^{(n)}}$, the conditional expected value $m(B^{(n)}, A^{(n)})$ of $\zeta_k^{(n)}$, given the prevailing bid and ask prices is independent of $n \in \mathbb{N}$. We assume that the mapping $m(\cdot,\cdot)$ is Lipschitz continuous.

Assumption 7. *The conditional expected value $m(B,A)$ of $\zeta_k^{(n)}$ depends in a Lipschitz continuous manner on the prevailing pair of bid and ask prices (B,A).*

1.1.8. State dynamics We are now ready to describe the full dynamics of the state sequence. To this end, we put

$$S_k^{(n)} = \left(B_k^{(n)}, A_k^{(n)}, v_{b,k}^{(n)}, v_{s,k}^{(n)}\right)$$

In terms of the indicator function $\mathbf{1}_k\left(S_k^{(n)}\right) := \left(\mathbf{1_A}\left(\phi_k(B_k^{(n)}, A_k^{(n)})\right), \ldots, \mathbf{1_H}\left(\phi_k(B_k^{(n)}, A_k^{(n)})\right)\right)'$ the dynamics of the state sequence $\{S_k^{(n)}\}$ is of the form

$$S_{k+1}^{(n)} = S_k^{(n)} + \mathcal{D}_k^{(n)}(S_k^{(n)})$$

if we define the random operator $\mathcal{D}_k^{(n)} : E \to E$ by

$$\mathcal{D}_k^{(n)}(S_k^{(n)}) := \mathbb{M}_k^{(n)}(S_k^{(n)}) \cdot \mathbf{1}_k(S_k^{(n)}) \tag{9}$$

where the matrix $\mathbb{M}_k^{(n)}(S_k^{(n)})$ equals

$$\begin{pmatrix} -\Delta x^{(n)} & \Delta x^{(n)} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \Delta x^{(n)} & -\Delta x^{(n)} & 0 & 0 \\ M_k^{(n),A} & M_k^{(n),B} & -\frac{\Delta v^{(n)}}{\Delta x^{(n)}} M_k^{(n),C} \cdot v_{b,k}^{(n)} & \frac{\Delta v^{(n)}}{\Delta x^{(n)}} M_k^{(n),D} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & M_k^{(n),E} & M_k^{(n),F} & -\frac{\Delta v^{(n)}}{\Delta x^{(n)}} M_k^{(n),G} \cdot v_{s,k}^{(n)} & \frac{\Delta v^{(n)}}{\Delta x^{(n)}} M_k^{(n),H} \end{pmatrix}.$$

Here, the entries referring to shifts in the volume density functions, due to best bid and ask price changes, are given by

$$\begin{aligned} M_k^{(n),A} &:= T_+^{(n)}\left(v_{b,k}^{(n)}\right) - v_{b,k}^{(n)}, \qquad & M_k^{(n),E} &:= T_+^{(n)}\left(v_{s,k}^{(n)}\right) - v_{s,k}^{(n)} \\ M_k^{(n),B} &:= T_-^{(n)}\left(v_{b,k}^{(n)}\right) - v_{b,k}^{(n)}, \qquad & M_k^{(n),F} &:= T_-^{(n)}\left(v_{s,k}^{(n)}\right) - v_{s,k}^{(n)} \end{aligned} \tag{10}$$

and the entries referring the volume changes, due to placement and cancelation of volume, are given by

$$M_{v,k}^{(n),I}(x) := \omega_k^I \sum_{j=-\infty}^{\infty} \mathbf{1}_{\{\pi_k^I \in [x_j^{(n)}, x_{j+1}^{(n)})\}}(x) \text{ for events I=}\mathbf{C}, \mathbf{D}, \mathbf{G}, \mathbf{H}. \tag{11}$$

Observing the dynamics in continuous time, we define

$$S^{(n)}(t) := S_k^{(n)} \quad \text{and} \quad \tau^{(n)}(t) := \tau_k^{(n)} \quad \text{for} \quad t \in [\tau_k^{(n)}, \tau_{k+1}^{(n)}). \tag{12}$$

Remark 4. Overall, the state and time dynamics of our models are driven by the random sequences $\{\varphi_k\}$ (event times), $\{\phi_k\}$ (event types), $\{\pi_k^I\}$ (placement/cancelation price levels) and $\{\omega_k^I\}$ (placed/canceled orders). The joint dynamics of all models can be defined in terms of suitable independent families

$$\kappa_k := \left\{ (\varphi_k(B,A), \phi_k(B,A))_{(A,B)\in\mathbb{R}^2}, (\pi_k^I, \omega_k^I)_{I=A,\dots,H} \right\} \quad (k \in \mathbb{N}_0)$$

of independent random variables. In particular, the process $\{(S^{(n)}(t), \tau^{(n)}(t))\}_{t\in[0,T]}$ $(n \in \mathbb{N})$ is adapted to the filtration

$$\mathcal{F}_k^{(n)} := \sigma\left(\kappa_s : 0 \le s \le \lfloor \frac{k}{\Delta t^{(n)}} \rfloor\right).$$

1.2. The main result Our main result is Theorem 1. It states that - with our choice of scaling - the order book dynamics can be described by a coupled ODE:PDE system when $n \to \infty$: the dynamics of the best bid and ask prices will be given in terms of an ODE, while that of the relative buy and sell volume densities will be given by the respective unique classical solution of a first order linear hyperbolic PDE with variable coefficients.

Theorem 1 **(Law of Large Numbers for LOBs)**. *Let $\{S^{(n)}\}_{n\ge1}$ be the sequence of continuous time processes defined in (12) and suppose that Assumptions 1 -7 hold. Then, for all $T > 0$ there exists a deterministic process $s : [0,T] \to E$ such that*

$$\lim_{n\to\infty} \sup_{t\in[0,T]} \|S^{(n)}(t) - s(t)\|_E = 0 \quad \textit{in probability.}$$

The process s is of the form $s(t) = \begin{pmatrix} \gamma(t) \\ v(\cdot,t) \end{pmatrix}$, where $\gamma(t) = \begin{pmatrix} b(t) \\ a(t) \end{pmatrix}$ is the vector of the best bid and ask prices at time $t \in [0,T]$ and $v(x,t) = \begin{pmatrix} v_b(x,t) \\ v_s(x,t) \end{pmatrix}$ denotes the vector of buy and sell volume densities at $t \in [0,T]$ relative to the best bid and ask price. In terms of the matrices

$$A(\cdot) := \begin{pmatrix} p^B(\cdot) - p^A(\cdot) & 0 \\ 0 & p^E(\cdot) - p^F(\cdot) \end{pmatrix}, \quad B(\cdot,x) := \begin{pmatrix} -p^C(\cdot)f^C(x) & 0 \\ 0 & -p^G(\cdot)f^G(x) \end{pmatrix}, \tag{13}$$

the vector

$$c(\cdot,x) := \begin{pmatrix} p^D(\cdot)f^D(x) \\ \\ p^H(\cdot)f^H(x) \end{pmatrix}, \tag{14}$$

and the function $m(\cdot,\cdot)$ that specifies the expected waiting time between two consecutive active order arrivals, the function γ is the unique solution to the 2-dimensional ODE system

$$\begin{cases} \frac{\mathrm{d}\gamma(t)}{\mathrm{d}t} = \frac{A(\gamma(t))}{m(\gamma(t))} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad t \in [0,T] \\ \gamma(0) = \begin{pmatrix} B_0 \\ A_0 \end{pmatrix} \end{cases} \tag{15}$$

and (v_b, v_s) is the unique non-negative bounded classical solution of the PDE

$$\begin{cases} v_t(t,x) = \frac{1}{m(\gamma(t))}\Big(A(\gamma(t))\, v_x(t,x) + B(\gamma(t),x)v(t,x) + c(\gamma(t),x)\Big), & (t,x) \in [0,T]\times\mathbb{R} \\ v(0,x) = v_0(0,x), & x \in \mathbb{R} \end{cases}. \tag{16}$$

REMARK 5. It is worth comparing our limiting model to that in [16] where a PDE limit for the LOB is obtained by modeling the dynamics of unexecuted orders using birth-death processes. Their limiting dynamics is less general than ours as the price process is constant in the limit. In our framework this corresponds to the case where $\Delta p^{(n)} = o\left(\frac{\Delta v^{(n)}}{\Delta x^{(n)}}\right)$ rather than $\Delta p^{(n)} = \mathcal{O}\left(\frac{\Delta v^{(n)}}{\Delta x^{(n)}}\right)$. In this case $A(.) \equiv 0$, and the limiting PDE simplifies to a family of ODEs. If $\Delta p^{(n)} = o\left(\frac{\Delta v^{(n)}}{\Delta x^{(n)}}\right)$, then our method easily extends to models where prices depend on volumes in the approximating models as in [16]. A general model with fully state dependent order dynamics has recently been established in [20].

The analysis of the limiting dynamics can be simplified by separating the randomness on the level of order arrival times from that of order types as shown in the following section. Subsequently, we give an explicit solution to the limiting PDE.

1.2.1. State and time separation For the continuous-time process $S^{(n)}$ we write

$$S^{(n)}(t) = \left(S_\gamma^{(n)}(t), S_v^{(n)}(t)\right)$$

where $S_\gamma^{(n)} \in \mathbb{R}^2$ describes the dynamics of bid and ask prices, and $S_v^{(n)}(t) \in L^2(\mathbb{R}) \times L^2(\mathbb{R})$ describes the dynamics of the buy and sell volume density functions. According to the following proposition the process can be expressed as the composition of a *state process* $\eta^{(n)}$ and a *time process* $\mu^{(n)}$. The proof follows from straightforward modifications of arguments given in Anisimov [4, p.108].

PROPOSITION 1 **(State and time separation)**. *The process $S^{(n)}$ can be expressed as the composition of a random state process*

$$\eta^{(n)}(t) = \left(\eta_\gamma^{(n)}(t), \eta_v^{(n)}(t)\right)$$

and a random time process $\mu^{(n)}$ as

$$S^{(n)}(t) = \eta^{(n)}\left(\mu^{(n)}(t) - \Delta t^{(n)}\right).$$

The state and time process is given by

$$\eta^{(n)}(t) := S_k^{(n)} \quad \textit{for} \quad t \in \left[t_k^{(n)}, t_{k+1}^{(n)}\right) \tag{17}$$

where $t_k^{(n)} := k \cdot \Delta t^{(n)}$ and

$$y^{(n)}(u) := \tau_k^{(n)} \quad \textit{for} \quad u \in \left[\tau_k^{(n)}, \tau_{k+1}^{(n)}\right), \tag{18}$$

respectively. The time-change $\mu^{(n)}$ is then defined in terms of $y^{(n)}$ as

$$\mu^{(n)}(t) := \inf\{u > 0 : y^{(n)}(u) > t\}.$$

The advantage of the state and time separation is that the processes $\eta^{(n)}$ and $\mu^{(n)}$ can be analyzed separately. In fact, we will show convergence in probability

$$\lim_{n\to\infty}\sup_{t\in[0,T]}\left\|\eta^{(n)}(t)-\eta(t)\right\|_E=0 \quad\text{and}\quad \lim_{n\to\infty}\sup_{t\in[0,T]}\left|\mu^{(n)}(t)-\mu(t)\right|=0$$

to limiting processes $\eta(t)=(\eta_\gamma(t),\eta_v(t))$ and $\mu(t)$. Since the state sequence takes values in the Hilbert space E, the time change theorem as proved in, e.g. Billingsley [8, p. 151], then implies that

$$\lim_{n\to\infty}\sup_{t\in[0,T]}\|S^{(n)}(t)-\eta(\mu(t))\|_E=0 \quad \text{in probability.}$$

In our model, bid and ask prices are sufficient statistics for the evolution of the order book. In particular, the limiting behavior of the sequences $\eta_\gamma^{(n)}$ and $\mu^{(n)}$ can be analyzed without reference to volumes. In Section 2 we prove the following proposition.

Proposition 2. *Let $\hat\gamma$ be the unique solution to the ODE*

$$\begin{cases}\frac{\mathrm{d}\hat\gamma(t)}{\mathrm{d}t}=A(\hat\gamma(t))\begin{pmatrix}1\\1\end{pmatrix}, & t\in(0,T]\\[2ex] \hat\gamma(0)=\begin{pmatrix}B_0\\A_0\end{pmatrix}.\end{cases} \qquad (19)$$

Then,

$$\lim_{n\to\infty}\sup_{0\le t\le T}\left|\eta_\gamma^{(n)}(t)-\hat\gamma(t)\right|=0 \quad \textit{in probability.}$$

Moreover, the sequence of processes $\mu^{(n)}$ satisfies

$$\lim_{n\to\infty}\sup_{0\le t\le T}\left|\mu^{(n)}(t)-\mu(t)\right|=0 \quad \textit{in probability.} \qquad \textit{where} \qquad \mu^{-1}(t)=\int_0^t m(\hat\gamma_u)du.$$

Once the limiting time-change process μ has been identified, what remains to finish the proof of Theorem 1, is to establish convergence of the volume processes $\eta_v^{(n)}$ to their deterministic limit. This will be achieved in Section 3 where we prove the following result.

Proposition 3. *Let $\widehat{u}$ be the unique classical solution of the PDE*

$$\begin{cases}\widehat{u}_t(x,t)=A\left(\widehat\gamma(t)\right)\widehat{u}_x(x,t)+B(x,\widehat\gamma(t))\widehat{u}(x,t)+c\left(x,\widehat\gamma(t)\right), & (x,t)\in\mathbb{R}\times[0,T]\\ \widehat{u}(x,0)=v_0(x), & x\in\mathbb{R}\end{cases}. \qquad (20)$$

Then

$$\lim_{n\to\infty}\sup_{0\le t\le T}\left\|\eta_v^{(n)}(t;\cdot)-\widehat{u}(t,\cdot)\right\|_{L^2}=0 \quad \textit{in probability.}$$

1.2.2. Explicit solution The PDE system (20) is coupled only through the limiting price dynamics. In particular, the equations for the buy and sell side can be solved independently. For the buy side PDE, we can write

$$\begin{cases}\frac{\partial u_b}{\partial t}=A_b(t)\frac{\partial u_b}{\partial x}+B_b(t,x)u_b+c_b(t,x)\\ u_b(0,x)=v_{b,0}(x)\end{cases} \qquad (21)$$

where $A_b(t) = p^B(\hat{\gamma}(t)) - p^A(\hat{\gamma}(t))$, $B_b(t,x) := -p^C(\hat{\gamma}(t))f^C(x)$, $c_b(t,x) := p^D(\hat{\gamma}(t))f^D(x)$. Using the method of characteristic curves, the PDE reduces to a family of ODEs; see [14, Chapter 3] for details. The characteristic equations for our buy side PDE read:

$$\begin{cases} \frac{\mathrm{d}x}{\mathrm{d}\tau} = -A_b(\tau) \\ x(0) = \xi \end{cases} \quad \text{and} \quad \begin{cases} \frac{\mathrm{d}\bar{u}_b}{\mathrm{d}\tau} = B_b(\tau, x(\tau))\bar{u}_b + c_b(\tau, x(\tau)) \\ \bar{u}_b(0,\xi) = v_{b,0}(\xi). \end{cases} \tag{22}$$

The solution to this ODE-system as a function of the state $\xi \in \mathbb{R}$ can be given in closed form:

$$\begin{aligned} x(t,\xi) &= \xi - \int_0^t A_b(t)\mathrm{d}t \\ \bar{u}_b(t,\xi) &= \exp\Big(\int_0^t B_b(u, x(u,\xi))du\Big)\Big(v_{b,0}(\xi) + \int_0^t \exp\Big(-\int_0^s B_b(u, x(u,\xi))du\Big)c_b(s, x(s,\xi))ds\Big). \end{aligned}$$

It describes the surface $\{(t,\xi) : u_b(t, x(t,\xi)) = \bar{u}_b(t,\xi)$ given $u_b(0,\xi) = v_{0,b}(\xi)\}$. The solution to the buy side PDE can be recovered from the solution to the ODE-system through

$$u_b(t,y) = \bar{u}_b\Big(t, y + \int_0^t A_b(s)ds\Big).$$

Due to our smoothness assumptions on the volume placement and cancelation functions it is not hard to verify that the solution is uniformly bounded with uniformly bounded first and second order derivatives with respect to the time and space variable. Moreover, since the function $v_{b,0}$ vanishes outside a compact interval (Assumption 1) and no orders are placed or canceled beyond a distance M from the best bid/ask price (Assumption 2), the function $u_b(t,\cdot)$ vanishes outside some compact interval $I(T)$ for all $t \in [0,T]$. Altogether, one has the following result.

Proposition 4. *Under the assumptions of Theorem 1, the PDE (16) has a unique solution u_b. The solution is uniformly bounded, with uniformly bounded first and second order derivatives with respect to both variables, and there exists an interval I such that $u_b(t,x) = 0$ for all $t \in [0,T]$ and $x \notin I$.*

1.3. A benchmark model with Poisson arrivals In this section we discus examples where the limiting dynamics can be given in closed form, provide simulation results that illustrate how our model could be used to approximate volume dynamics, and illustrate how our model could be applied to portfolio liquidation problems.

Let us assume that orders arrive according to independent Poisson processes with smooth, price-dependent rate functions $\lambda^A(\cdot,\cdot), ..., \lambda^H(\cdot,\cdot)$ in the benchmark model $n = 1$. Standard arguments yield:

$$m(\cdot,\cdot) = \frac{1}{\lambda^A(\cdot,\cdot) + \cdots + \lambda^H(\cdot,\cdot)}.$$

To simplify the analysis we normalize the arrival rates so that $m(\cdot,\cdot) \equiv 1$. In the n:th model we scale the arrival rates of passive orders by $\frac{1}{\Delta t^{(n)}}$ and those of active orders by $\frac{1}{\Delta x^{(n)}}$. The arrival rates in the n:th model thus satisfy:

$$\lambda^{(n),I}(\cdot,\cdot) = \frac{\lambda^I(\cdot,\cdot)}{\Delta x^{(n)}} \quad \text{for} \quad I = \mathbf{A}, \mathbf{B}, \mathbf{E}, \mathbf{F}$$

and

$$\lambda^{(n),I}(\cdot,\cdot) = \frac{\lambda^I(\cdot,\cdot)}{\Delta t^{(n)}} \quad \text{for} \quad I = \mathbf{C}, \mathbf{D}, \mathbf{G}, \mathbf{H}.$$

Then, the probability of the next event being an active order is of the order $\Delta p^{(n)} = \frac{\Delta t^{(n)}}{\Delta x^{(n)}}$ and:

$$p^I(\cdot,\cdot) = \frac{\lambda^I(\cdot,\cdot)}{\lambda^A(\cdot,\cdot)+\lambda^B(\cdot,\cdot)+\lambda^E(\cdot,\cdot)+\lambda^F(\cdot,\cdot)} \quad \text{for} \quad I = \mathbf{A},\mathbf{B},\mathbf{E},\mathbf{F}$$
$$p^I(\cdot,\cdot) = \frac{\lambda^I(\cdot,\cdot)}{\lambda^C(\cdot,\cdot)+\lambda^D(\cdot,\cdot)+\lambda^G(\cdot,\cdot)+\lambda^H(\cdot,\cdot)} \quad \text{for} \quad I = \mathbf{C},\mathbf{D},\mathbf{G},\mathbf{H}.$$

1.3.1. Price dynamics Of course, the spread

$$\mathfrak{S}(t) := a(t) - b(t)$$

should be non-negative at all times. This can easily be achieved if we require $p^A(\cdot,\cdot) = p^E(\cdot,\cdot) = 0$ for $\mathfrak{S}(t) = 0$. For most applications it would in fact be sufficient to assume that the price dynamics depends on the best bid/ask price only through the spread. If the stationarity condition

$$\lambda^A - \lambda^B = \lambda^E - \lambda^F = 0$$

holds, then the spread is constant: $\mathfrak{S}(t) \equiv \mathfrak{S}(0)$.

The following is a simple example where the spread is initially positive, then becomes zero, and eventually opens again.

Example 1. Assume that the order arrival rates take the form

$$\begin{cases} \lambda^A(b(t),a(t)) & = \lambda^B(b(t),a(t)) \\ \lambda^E(b(t),a(t)) - \lambda^F(b(t),a(t)) & = 2\sqrt{a(t)-B_0}, \quad t\in[0,T] \end{cases}.$$

Then, the bid price is constant, $b(t) \equiv B_0$, and $a(t) = B_0 + \left(t - \frac{T}{2}\right)^2$ for $t \in [0,T]$; see Figure 4.

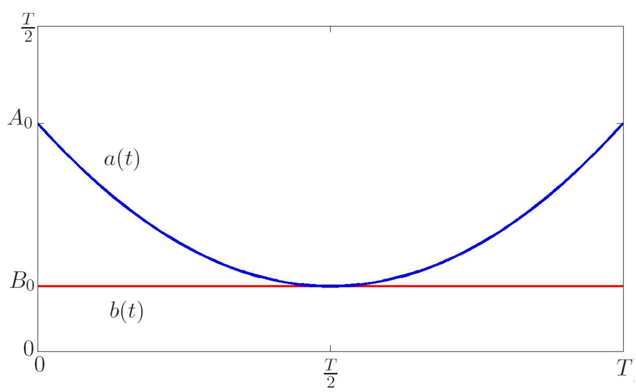

Figure 4. Bid/ask price dynamics of Example 1.

Next, we consider an example where the spread settles to a stationary level.

Example 2. Assume that the active order arrival rates take the form:

$$\lambda^A(b(t),a(t)) := \frac{1+\mu}{2}\exp(-\mathfrak{S}(t)^+)$$
$$\lambda^E(b(t),a(t)) := \frac{1-\mu}{2}\exp(-\mathfrak{S}(t)^+)$$
$$\lambda^B(b(t),a(t)) := \frac{1-\mu}{2}\Big(1-\exp(-\mathfrak{S}(t)^+)\Big)$$
$$\lambda^F(b(t),a(t)) := \frac{1+\mu}{2}\Big(1-\exp(-\mathfrak{S}(t)^+)\Big)$$

for some $\mu \in (0,1)$ where $x^+ := \max\{0,x\}$ denotes the positive part of $x \in \mathbb{R}$. In particular, the probability of spread placements increases with the spread, no spread placements occur if $\mathfrak{S}(t) = 0$, and the limiting price dynamics is (cf. Figure 5):

$$\frac{db(t)}{dt} = \frac{1-\mu}{2} - e^{-\mathfrak{S}(t)^+} \quad \text{and} \quad \frac{da(t)}{dt} = e^{-\mathfrak{S}(t)^+} - \frac{1+\mu}{2}. \tag{23}$$

The probability of a price increase, respectively, decrease can also be approximated in terms of the arrival rates. If the next event is an active order arrival on the buy side, then

$$\lim_{n\to\infty} \mathbb{P}[B^{(n)}(t) = B^n(t-) - \Delta x^{(n)}] = \frac{\lambda^A}{\lambda^A + \lambda^B},$$
$$\lim_{n\to\infty} \mathbb{P}[B^{(n)}(t) = B^n(t-) + \Delta x^{(n)}] = \frac{\lambda^B}{\lambda^A + \lambda^B}.$$

The limiting probabilities that the next price change is an increase/decrease of the best bid/ask price as well as unconditional probabilities of price changes can be computed analogously. Clearly, these probabilities increase in the respective rates. A further possible application of our model includes estimations of the expected time-to-fill of a limit order in the original model. This question has been studied by, e.g. Lo et al [23]. Our model yields an approximation for that time in terms of the first time the limiting price process hits the placement price level of the limit order.

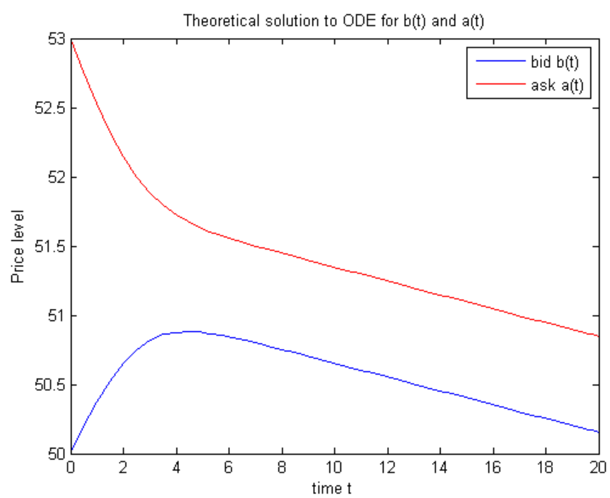

FIGURE 5. Bid/ask price dynamics of Example 3 for $\mu = 0.1; b(0) = 50; a(0) = 53$.

1.3.2. Volume dynamics We now turn to the volume dynamics. When the limiting price is constant, then the explicit solution to the bid-side volume density function given in (22) simplifies to

$$u_b(t,y) = e^{tB_b(y)} \left(v_{b,0}(y) + \frac{c_b(y)}{B_b(y)} \left[1 - e^{-tB_b(y)} \right] \right) \tag{24}$$

where we write $B_b(y)$ and $c_b(y)$ for $B_b(\gamma(0),y)$ and $c_b(\gamma(0),y)$, respectively. From this we see that the stationary solution is

$$u_b(t,y) = -\frac{c_b(y)}{B_b(y)} = \frac{p^D(\gamma(0))f^D(y)}{p^C(\gamma(0))f^C(y)} \equiv \frac{p^D}{p^C} \cdot \frac{f^D(y)}{f^C(y)}. \tag{25}$$

In particular, the volume at the price level y increases in the order arrival rates and the "probabilities" $f^D(y)$ with which an order is placed at that level when a placement occurs. Analogously, the volume decreases in the cancellation rate and $f^C(y)$.

The following is a simple example where the expected volume dynamics is non-stationary but can still be given in closed form.

Example 3. Assume that

$$\lambda^A(b(t),a(t)) - \lambda^B(b(t),a(t)) = \lambda^E(b(t),a(t)) - \lambda^F(b(t),a(t)) = 1.$$

Then $b(t) = b_0 + t$ and $a(t) = b(t) + \mathfrak{S}(0)$. Assume moreover that $b_0 = 0$ and that the buy side passive order arrival rates depend only on the spread:

$$\lambda^C(b(t),a(t)) = \lambda^D(b(t),a(t)) = a(t) - b(t).$$

Thus, $p^{C,D}(\gamma(t)) \equiv p^{C,D}$. Let us further assume (ignoring the fact that our density functions need to be defined on compact intervals for simplicity) that

$$f^C(x) \equiv \frac{1}{p^C} \quad \text{and} \quad f^D(x) = \frac{e^{-x}}{p^D}.$$

Then,

$$B_b(\gamma(t),y) \equiv -1 \quad \text{and} \quad c_b(\gamma(t),y) = e^{-y}.$$

We compute from the general solution formula:

$$u_b(t,x) = e^{-t}u_b(0,x) + e^{-y}(1-e^{-t}).$$

2. Convergence of bid/ask prices According to Proposition 1, the process $S^{(n)}$ can be represented in terms of a composition of a state process $\eta^{(n)}$ that jumps at deterministic times $\{t_k^{(n)}\}$ and a time process $\mu^{(n)}$ that accounts for the random arrival times. Prices change less frequently at times $\{\sigma_k^{(n)}\}$. This suggests to introduce a second time scale - which will be referred to as *active order time* - defined by

$$s_k^{(n)} := k \cdot \Delta x^{(n)}$$

along which to scale the price process. In order to make this more precise, let us denote by $\mathcal{D}_{\gamma,k}^{(n)}$ the restriction of the operator $\mathcal{D}_k^{(n)}$ to the price component of the state sequence and put

$$\widehat{\mathcal{D}}_{\gamma,k}^{(n)} := \sum_{l=\sigma_{k-1}^{(n)}+1}^{\sigma_k^{(n)}-1} \mathcal{D}_{\gamma,l}^{(n)} = \mathcal{D}_{\gamma,\sigma_k^{(n)}}^{(n)}$$

where the second equality follows from the fact that prices do not change between the times $\sigma_{k-1}^{(n)}+1$ and $\sigma_k^{(n)}-1$. Furthermore, we introduce the family of continuous time stochastic processes $\widehat{\eta}_\gamma^{(n)}$ defined by

$$\widehat{\eta}^{(n)}(t) := \widehat{\eta}_k^{(n)} \quad \text{for } t \in [s_k^{(n)}, s_{k+1}^{(n)})$$

where

$$\begin{cases} \widehat{\eta}_{\gamma,k+1}^{(n)} & := \widehat{\eta}_{\gamma,k}^{(n)} + \widehat{\mathcal{D}}_{\gamma,k}^{(n)}(\widehat{\eta}_k^{(n)}) \\ \\ \widehat{\eta}_0^{(n)} & = \begin{pmatrix} B_0^{(n)} \\ A_0^{(n)} \end{pmatrix}. \end{cases} \tag{26}$$

The quantity $\widehat{\eta}_{\gamma,k}^{(n)}$ describes the state of the price process after the k:th price change. The following lemma shows that the process $\eta_\gamma^{(n)}$, evolving on the level of event time, and the process $\widehat{\eta}_\gamma^{(n)}$, evolving on the level of active order time, are indistinguishable in the limit when $n \to \infty$.

LEMMA 1. *For any $T>0$ and $\epsilon>0$, it holds that*

$$\lim_{n\to\infty}\mathbb{P}\left[\sup_{0\le t\le T}\left|\sum_{k=0}^{t/\Delta x^{(n)}}\widehat{\mathcal{D}}^{(n)}_{\gamma,k}-\sum_{k=0}^{t/\Delta t^{(n)}}\mathcal{D}^{(n)}_{\gamma,k}\right|>\epsilon\right]=0.$$

Proof. By construction, the two sums $\sum_{k=0}^{\lfloor t/\Delta x^{(n)}\rfloor}\widehat{\mathcal{D}}^{(n)}_{\gamma,k}$ and $\sum_{k=0}^{\lfloor t/\Delta t^{(n)}\rfloor}\mathcal{D}^{(n)}_{\gamma,k}$ have the same expected value for any $t\in[0,T]$:

$$\sum_{k=0}^{\lfloor t/\Delta x^{(n)}\rfloor}\mathbb{E}\widehat{\mathcal{D}}^{(n)}_{\gamma,k}=\sum_{k=0}^{\lfloor t/\Delta t^{(n)}\rfloor}\mathbb{E}\mathcal{D}^{(n)}_{\gamma,k}.$$

As a result, it is enough to prove that

$$\lim_{n\to\infty}\mathbb{P}\left[\sup_{0\le t\le T}\left|\sum_{k=0}^{\lfloor t/\Delta x^{(n)}\rfloor}\left\{\widehat{\mathcal{D}}^{(n)}_{\gamma,k}-\mathbb{E}\widehat{\mathcal{D}}^{(n)}_{\gamma,k}\right\}\right|>\frac{\epsilon}{2}\right]=\lim_{n\to\infty}\mathbb{P}\left[\sup_{0\le t\le T}\left|\sum_{k=0}^{\lfloor t/\Delta t^{(n)}\rfloor}\left\{\mathcal{D}^{(n)}_{\gamma,k}-\mathbb{E}\mathcal{D}^{(n)}_{\gamma,k}\right\}\right|>\frac{\epsilon}{2}\right]=0.$$

The random variables

$$\frac{1}{\Delta x^{(n)}}\left\{\widehat{\mathcal{D}}^{(n)}_{\gamma,k}-\mathbb{E}\widehat{\mathcal{D}}^{(n)}_{\gamma,k}\right\},\quad k=0,\dots,\lfloor T/\Delta x^{(n)}\rfloor,\quad n\in\mathbb{N}$$

and

$$\frac{1}{\Delta t^{(n)}}\left\{\mathcal{D}^{(n)}_{\gamma,k}-\mathbb{E}\mathcal{D}^{(n)}_{\gamma,k}\right\},\quad k=0,\dots,\lfloor T/\Delta t^{(n)}\rfloor,\quad n\in\mathbb{N}$$

form triangular martingale difference arrays in the sense of Definition 2 (Appendix A) with respect to the filtrations $\{\mathcal{F}_{\sigma^{(n)}_k}\}_{k\in\mathbb{N}}$ and $\{\mathcal{F}_k\}_{k\in\mathbb{N}}$, respectively. A direct computation shows that they are uniformly L^2-bounded. Thus, it follows from Theorem 2 in Appendix A that for all $\beta>\frac{1}{2}$:

$$\lim_{n\to\infty}\mathbb{P}\left[\sup_{0\le m\le\lfloor\frac{T}{\Delta x^{(n)}}\rfloor}\frac{1}{\Delta x^{(n)}}\left|\sum_{k=0}^{m}\left\{\widehat{\mathcal{D}}^{(n)}_{\gamma,k}-\mathbb{E}\widehat{\mathcal{D}}^{(n)}_{\gamma,k}\right\}\right|\ge\frac{\epsilon}{2}\left(\frac{T}{\Delta x^{(n)}}\right)^{\beta}\right]=0$$

as well as

$$\lim_{n\to\infty}\mathbb{P}\left[\sup_{0\le m\le\lfloor\frac{T}{\Delta t^{(n)}}\rfloor}\frac{1}{\Delta t^{(n)}}\left|\sum_{k=0}^{m}\left\{\mathcal{D}^{(n)}_{\gamma,k}-\mathbb{E}\mathcal{D}^{(n)}_{\gamma,k}\right\}\right|\ge\frac{\epsilon}{2}\left(\frac{T}{\Delta t^{(n)}}\right)^{\beta}\right]=0.$$

Choosing $\beta\in(\frac{1}{2},1)$ and multiplying the inequalities in the above probabilities by $\Delta x^{(n)}$ and $\Delta t^{(n)}$, respectively, proves the assertion. □

Let $\widehat{\gamma}$ be the solution to the ODE (19) and consider the discretization $\widehat{\gamma}^{(n)}_k:=\widehat{\gamma}(s^{(n)}_k)$. The next lemma shows that the sequence of expected price processes $\widetilde{\gamma}^{(n)}$ defined by

$$\widetilde{\gamma}^{(n)}(t):=\widetilde{\gamma}^{(n)}_k\quad\text{for } t\in[s^{(n)}_k,s^{(n)}_{k+1})$$

where

$$\begin{cases}\widetilde{\gamma}^{(n)}_{k+1} & :=\widetilde{\gamma}^{(n)}_k+\mathbb{E}\left[\widehat{\mathcal{D}}^{(n)}_{\gamma,k}(\widetilde{\gamma}^{(n)}_k)\right]\\[2ex] \widetilde{\gamma}^{(n)}_0 & =\begin{pmatrix}B^{(n)}_0\\ A^{(n)}_0\end{pmatrix}\end{cases}\tag{27}$$

converges uniformly to $\widehat{\gamma}$ on compact time intervals. The proof is standard; we give it merely for completeness.

LEMMA 2. *For any $T>0$*

$$\sup_{t\in[0,T]} |\widetilde{\gamma}^{(n)}(t)-\widehat{\gamma}(t)| = \mathcal{O}(\Delta x^{(n)}).$$

Proof. Active orders change prices by one tick. Moreover, by Assumption 5, the conditional probabilities of an active order being a market buy/sell order or limit buy/sell order placement in the spread are independent of $n\in\mathbb{N}$. Hence

$$\begin{aligned}\mathbb{E}\left[\widehat{\mathcal{D}}^{(n)}_{\gamma,k}(\widehat{\gamma}^{(n)}_k)\right] &= \Delta x^{(n)}\cdot\begin{pmatrix} p^B(\widehat{\gamma}^{(n)}_k)-p^A(\widehat{\gamma}^{(n)}_k)\\ p^E(\widehat{\gamma}^{(n)}_k)-p^F(\widehat{\gamma}^{(n)}_k)\end{pmatrix}\\ &= \Delta x^{(n)}\cdot\left\{A(\widehat{\gamma}^{(n)}_k)\begin{pmatrix}1\\1\end{pmatrix}\right\}.\end{aligned} \tag{28}$$

Thus, the sequence $\widehat{\gamma}^{(n)}$ defines a special case of the classical Euler scheme for the ODE (15) and hence converges uniformly to its unique solution, see e.g. Hairer et al. [19, Theorem 7.3], with rate $\Delta x^{(n)}$. □

We are now ready to prove convergence in probability of the bid and ask prices.

PROOF OF PROPOSITION 2.

a) We first consider the convergence of the state process $\eta^{(n)}_\gamma$ and claim that

$$\lim_{n\to\infty}\sup_{t\in[0,T]}|\eta^{(n)}_\gamma(t)-\widehat{\gamma}(t)|\to 0,\quad \text{in probability.} \tag{29}$$

In view of Lemma 1, we can write

$$\begin{aligned}\eta^{(n)}_\gamma(t) &= \widehat{\eta}^{(n)}_\gamma(t)\\ &= \gamma^{(n)}(0)+\sum_{k=0}^{\lfloor t/\Delta x^{(n)}\rfloor}\widehat{\mathcal{D}}^{(n)}_{\gamma,k}\left(\widehat{\eta}^{(n)}_{\gamma,k}\right)\\ &= \gamma^{(n)}(0)+\sum_{k=0}^{\lfloor t/\Delta x^{(n)}\rfloor}\mathbb{E}\left[\widehat{\mathcal{D}}^{(n)}_{\gamma,k}(\widehat{\eta}^{(n)}_{\gamma,k})\right]+\sum_{k=0}^{\lfloor t/\Delta x^{(n)}\rfloor}\left(\widehat{\mathcal{D}}^{(n)}_{\gamma,k}(\widehat{\eta}^{(n)}_{\gamma,k})-\mathbb{E}\left[\widehat{\mathcal{D}}^{(n)}_{\gamma,k}(\widehat{\eta}^{(n)}_{\gamma,k})\right]\right)\end{aligned}$$

up to some random additive constant that vanishes almost surely uniformly in $t\in[0,T]$ as $n\to\infty$. Adding and subtracting the sequence $\widetilde{\gamma}^{(n)}$ yields (again up to a vanishing additive constant):

$$\begin{aligned}\left|\eta^{(n)}_\gamma(t)-\widehat{\gamma}(t)\right| \le{}& \left|\widetilde{\gamma}^{(n)}(t)-\widehat{\gamma}(t)\right|\\ &+\left|\sum_{k=0}^{\lfloor t/\Delta x^{(n)}\rfloor}\mathbb{E}\left[\widehat{\mathcal{D}}^{(n)}_{\gamma,k}(\widehat{\eta}^{(n)}_{\gamma,k})\right]-\widetilde{\gamma}^{(n)}(t)\right|\\ &+\left|\sum_{k=0}^{\lfloor t/\Delta x^{(n)}\rfloor}\left(\widehat{\mathcal{D}}^{(n)}_{\gamma,k}(\widehat{\eta}^{(n)}_{\gamma,k})-\mathbb{E}\left[\widehat{\mathcal{D}}^{(n)}_{\gamma,k}(\widehat{\eta}^{(n)}_{\gamma,k})\right]\right)\right|.\end{aligned}$$

For the first term, we deduce from Lemma 2 that

$$\sup_{t\in[0,T]}|\widetilde{\gamma}^{(n)}(t)-\widehat{\gamma}(t)| = \mathcal{O}(\Delta t^{(n)}). \tag{30}$$

For the second term we use the Lipschitz continuity of the event probabilities $p^I(\cdot,\cdot)$ in order to establish the existence of a constant $L_\gamma > 0$ such that:

$$\begin{aligned}\left|\sum_{k=0}^{\lfloor t/\Delta x^{(n)}\rfloor} \mathbb{E}\left[\widehat{\mathcal{D}}^{(n)}_{\gamma,k}(\widetilde{\eta}^{(n)}_{\gamma,k})\right] - \widetilde{\gamma}^{(n)}(t)\right| &= \left|\sum_{k=0}^{\lfloor t/\Delta x^{(n)}\rfloor} \mathbb{E}\left[\widehat{\mathcal{D}}^{(n)}_{\gamma,k}(\widehat{\eta}^{(n)}_{\gamma,k})\right] - \mathbb{E}\left[\widehat{\mathcal{D}}^{(n)}_{\gamma,k}(\widehat{\gamma}^{(n)}_{k})\right]\right| \\ &\leq \sum_{k=0}^{\lfloor t/\Delta x^{(n)}\rfloor} \left|\mathbb{E}\left[\widehat{\mathcal{D}}^{(n)}_{\gamma,k}(\widehat{\eta}^{(n)}_{\gamma,k})\right] - \mathbb{E}\left[\widehat{\mathcal{D}}^{(n)}_{\gamma,k}(\widehat{\gamma}^{(n)}_{k})\right]\right| \\ &\leq \Delta x^{(n)} \cdot L_\gamma \cdot \sum_{k=0}^{\lfloor t/\Delta x^{(n)}\rfloor} \left|\widehat{\eta}^{(n)}_{\gamma,k} - \widehat{\gamma}^{(n)}_{k}\right|.\end{aligned}$$

The third term corresponds to the noise-term of the price process. For each $n \in \mathbb{N}$, the sequence

$$y^n_k := \widehat{\mathcal{D}}^{(n)}_{\gamma,k}(\widehat{\eta}^{(n)}_{\gamma,k}) - \mathbb{E}\left[\widehat{\mathcal{D}}^{(n)}_{\gamma,k}(\widehat{\eta}^{(n)}_{\gamma,k})\right], \quad k = 0, ..., \lfloor T/\Delta x^{(n)}\rfloor$$

is a martingale difference sequence. A direct computation shows that

$$\sup_{n,k} \mathbb{E}|y^n_k|^2 \leq C \cdot \left(\Delta x^{(n)}\right)^2.$$

Hence, the law of large numbers for triangular martingale difference arrays (Theorem 2 and Corollary 1) implies

$$\lim_{n\to\infty} \mathbb{P}\left[\sup_{0\leq m\leq T/\Delta x^{(n)}} |\sum_{k=0}^{m} y^n_k| > 0\right] = 0,$$

just as in the proof of Lemma 1. Thus, using Lemma 1 again, we see that

$$\left|\eta^{(n)}_\gamma(t) - \widehat{\gamma}(t)\right| = \left|\widehat{\eta}^{(n)}_\gamma(t) - \widehat{\gamma}(t)\right| \leq \Delta x^{(n)} \cdot L_\gamma \cdot \sum_{k=0}^{\lfloor t/\Delta x^{(n)}\rfloor} \left|\widehat{\eta}^{(n)}_{\gamma,k} - \widehat{\gamma}^{(n)}_{k}\right| + o(1) \quad \text{in probability}$$

for some additive term of order $o(1)$ uniform in $t \in [0,T]$. As a result, (29) follows from an application of Gronwall's lemma along with Lemma 1.

b) Let us now consider the cumulative "active order time process"

$$\begin{aligned}y^{(n)}(t) &= \sum_{k=0}^{\lfloor t/\Delta x^{(n)}\rfloor} \zeta^{(n)}\left(\widehat{\eta}^{(n)}_{\gamma,k}\right) \cdot \Delta x^{(n)} \\ &= \Delta x^{(n)} \cdot \left\{\sum_{k=0}^{\lfloor t/\Delta x^{(n)}\rfloor} \mathbb{E}\left[\zeta^{(n)}\left(\widehat{\eta}^{(n)}_{\gamma,k}\right)\right] + \sum_{k=0}^{\lfloor t/\Delta x^{(n)}\rfloor} \left(\zeta^{(n)}\left(\widehat{\eta}^{(n)}_{\gamma,k}\right) - \mathbb{E}\left[\zeta^{(n)}\left(\widehat{\eta}^{(n)}_{\gamma,k}\right)\right]\right)\right\} \\ &= \Delta x^{(n)} \sum_{k=0}^{\lfloor t/\Delta x^{(n)}\rfloor} m\left(\widehat{\eta}^{(n)}_{\gamma,k}\right) + \Delta x^{(n)} \cdot \sum_{k=0}^{\lfloor t/\Delta x^{(n)}\rfloor} \left(\zeta^{(n)}\left(\widehat{\eta}^{(n)}_{\gamma,k}\right) - \mathbb{E}\left[\zeta^{(n)}\left(\widehat{\eta}^{(n)}_{\gamma,k}\right)\right]\right).\end{aligned} \tag{31}$$

By the above established uniform convergence of $\widehat{\eta}^{(n)}_\gamma$ to $\widehat{\gamma}$ in probability and because the function m is Lipschitz continuous, the first sum converges to the function

$$y(t) = \int_0^t m(\widehat{\gamma}(u))\mathrm{d}u. \tag{32}$$

Applying the same arguments as above to the martingale difference sequences

$$\zeta^{(n)}\left(\widehat{\eta}^{(n)}_{\gamma,k}\right) - \mathbb{E}\left[\zeta^{(n)}_k\left(\widehat{\eta}^{(n)}_{\gamma,k}\right)\right], \quad k = 0, ..., \lfloor \frac{t}{\Delta x^{(n)}} \rfloor$$

we see that the second term vanishes uniformly in $t \in [0,T]$ in probability. Thus,

$$\lim_{n\to\infty} \sup_{t\in[0,T]} |y^{(n)}(t) - y(t)| = 0 \quad \text{in probability.}$$

Since $y^{(n)}$ and y are increasing functions, their inverses $\mu^{(n)}$ and μ exist. By continuity

$$\sup_{t\in[0,T]} |\mu^{(n)}(t) - \mu(t)| \to 0 \quad \text{in probability as } n \to \infty$$

and

$$\mu'(t) = \left(y^{-1}(t)\right)' = \frac{1}{y'(y^{-1}(t))} = \frac{1}{m(\mu(t))} = \frac{1}{m\left(\widehat{\gamma}(\mu(t))\right)}. \tag{33}$$

Since both the state and the time process converge, we conclude from the time change theorem that

$$\sup_{t\in[0,T]} \left|\eta^{(n)}_\gamma(t) - \gamma(t)\right| \to 0 \quad \text{in probability } n \to \infty,$$

where $\gamma(t) = \widehat{\gamma}(\mu(t))$ and

$$\gamma'(t) = \widehat{\gamma}'(\mu(t)) \cdot \mu'(t) = \frac{A(\widehat{\gamma}(\mu(t)))}{m(\widehat{\gamma}(\mu(t)))}\begin{pmatrix}1\\1\end{pmatrix} = \frac{A(\gamma(t))}{m(\gamma(t))}\begin{pmatrix}1\\1\end{pmatrix}.$$

□

3. Convergence of volume densities In this section we prove Proposition 3. To this end, we denote by $\mathcal{D}^{(n)}_{v,k}(\cdot,\cdot)$ the restriction of the operator $\mathcal{D}^{(n)}_k$ to $L^2 \times L^2$, i.e the restriction of $\mathcal{D}^{(n)}_k$ to the volume components of the state process. We need to show that the sequence $\{\eta^{(n)}_v\}_{n\in\mathbb{N}}$ of $L^2 \times L^2$-valued step-functions defined recursively by

$$\eta^{(n)}_v(t,\cdot) := \eta^{(n)}_{v,k} \quad \text{for } t \in [t^{(n)}_k, t^{(n)}_{k+1}) \tag{34}$$

where

$$\begin{cases} \eta^{(n)}_{v,k+1} & := \eta^{(n)}_k + \mathcal{D}^{(n)}_{v,k}\left(\eta^{(n)}_{\gamma,k}, \eta^{(n)}_{v,k}\right) \\ \\ \eta^{(n)}_{v,0} & := v^{(n)}_0 \end{cases} \tag{35}$$

converges in probability in L^2 to the unique solution of the PDE (20). We will show convergence in several steps. In a first step, we find a convergent discretization scheme of the PDE that is coherent with the order book dynamics. Subsequently, we link this scheme to the expected dynamics of the volume densities.

3.1. A numerical scheme for the limiting PDE For any $n \in \mathbb{N}$, the scaling parameters $\Delta x^{(n)}$ and $\Delta t^{(n)}$ define a grid $\{(t_k^{(n)}, x_k^{(n)})\}$ on $[0,T] \times \mathbb{R}$ through $t_k^{(n)} = k \cdot \Delta t^{(n)}$ $(k \in \mathbb{N}_0)$ and $x_j^{(n)} = j \cdot \Delta x^{(n)}$ $(j \in \mathbb{Z})$. In a first step, we approximate the unique solution $\widehat{u} : [0,T] \times \mathbb{R} \to \mathbb{R}^2$ to (20) by a sequence of grid-point functions $\widehat{u}^{(n)} : [0,T] \times \mathbb{R} \to \mathbb{R}^2$. To this end, we put

$$A_R(t) := \begin{pmatrix} p^B(\hat{\gamma}(t)) & 0 \\ 0 & p^E(\hat{\gamma}(t)) \end{pmatrix}, \quad A_L(t) := \begin{pmatrix} p^A(\hat{\gamma}(t)) & 0 \\ 0 & p^F(\hat{\gamma}(t)) \end{pmatrix}$$

and

$$F(t,x) := B(\hat{\gamma}(t), x), \quad g(t,x) := c(\hat{\gamma}(t), x).$$

Furthermore, we introduce operators $\mathcal{H}_t^{(n)}$ that act on $v \in L^2$ according to

$$\begin{aligned} \mathcal{H}_t^{(n)}(v) := v &+ \Delta p^{(n)} \cdot A_R(t) \left[v(\cdot + \Delta x^{(n)}) - v(\cdot) \right] \\ &+ \Delta p^{(n)} \cdot A_L(t) \left[v(\cdot - \Delta x^{(n)}) - v(\cdot) \right] \\ &+ \Delta v^{(n)} \cdot (1 - \Delta p^{(n)}) \cdot \left[F(t,\cdot) \cdot v(\cdot) + g(t,\cdot) \right]. \end{aligned}$$

The sequence of grid-point approximations is then defined recursively by

$$\widehat{u}^{(n)}(t,\cdot) := \widehat{u}_k^{(n)} \quad \text{for} \quad t \in [t_k^{(n}, t_{k+1}^{(n)}) \tag{36}$$

where

$$\begin{cases} \widehat{u}_{k+1}^{(n)} & = \mathcal{H}_{t_k^{(n)}}^{(n)}(\widehat{u}_k^{(n)}) \\ \\ \widehat{u}_0^{(n)} & = v_0^{(n)}. \end{cases} \tag{37}$$

The sequence of step-functions $\{\widehat{u}^{(n)}\}$ essentially describes a discretized limiting volume dynamics of the order book. We benchmark this dynamics against the expected pre-limit volume dynamics when prices are replaced by their limiting dynamics. More precisely, we introduce another sequence of step functions $u^{(n)} : [0,T] \times \mathbb{R} \to \mathbb{R}$ by

$$u^{(n)}(t,\cdot) := u_k^{(n)} \quad \text{for} \quad t \in [t_k^{(n}, t_{k+1}^{(n)}) \tag{38}$$

where

$$\begin{cases} u_{k+1}^{(n)} & = u_k^{(n)} + \mathbb{E}\left[\mathcal{D}_{v,k}^{(n)} \left(\hat{\gamma}(t_k^{(n)}), u_k^{(n)} \right) \right] \\ \\ u_0^{(n)} & := v_0^{(n)}. \end{cases} \tag{39}$$

In a first step, we are now going to show that the grid-point functions $\widehat{u}^{(n)}$ approximate the solution $\widehat{u}$ of our PDE. Subsequently, we show that the PDE can as well be approximated by the functions $u^{(n)}$.

Proposition 5 **(Convergence of the numerical scheme)**. *Assume that the assumptions of Theorem 1 hold. Then, the processes $\widehat{u}^{(n)}$ define a convergent finite difference scheme of the PDE (20), i.e.*

$$\sup_{t \in [0,T]} \|\widehat{u}^{(n)}(t,\cdot) - \widehat{u}(t,\cdot)\|_{L^2} \to 0 \quad \text{as } n \to \infty. \tag{40}$$

Proof. For $t \in [0,T]$ the local truncation error associated with the grid-point approximations $\widehat{u}^{(n)}$ is defined as

$$\mathcal{L}^{(n)}(t,x) := \frac{1}{\Delta t^{(n)}}\Big(\widehat{u}(t+\Delta t^{(n)},x) - \mathcal{H}_t^{(n)}(\widehat{u}(t,\cdot))(x)\Big). \tag{41}$$

Smoothness of the solution $\widehat{u}$ (bounded with uniformly bounded first and second order derivatives) along with Assumption 6 implies that the following estimate holds uniformly in $t \in [0,T]$ and $x \in \mathbb{R}$:

$$\begin{aligned}
\mathcal{L}^{(n)}(t,x) = \frac{1}{\Delta t^{(n)}}\Big(&\widehat{u}(t+\Delta t^{(n)},x) - \widehat{u}(t,x) - \Delta p^{(n)}\cdot A_R(t)(\widehat{u}(t,x+\Delta x^{(n)}) - \widehat{u}(t,x)) \\
&- \Delta p^{(n)}\cdot A_L(t)(\widehat{u}(t,x-\Delta x^{(n)}) - \widehat{u}(t,x)) \\
&- \Delta v^{(n)}\cdot(1-\Delta p^{(n)})\cdot[F(t,x)\widehat{u}(t,x) - g(t,x)]\Big) \\
= \frac{1}{\Delta t^{(n)}}\Big(&\widehat{u}_t(t,x)\Delta t^{(n)} + o(\Delta t^{(n)}) - \Delta p^{(n)}\cdot A_R(\widehat{u}_x(t,x)\Delta x^{(n)} + o(\Delta x^{(n)})) \\
&- \Delta p^{(n)}\cdot A_L(-\widehat{u}_x(t,x))\Delta x^{(n)} + o(\Delta x^{(n)})) \\
&- \Delta v^{(n)}\cdot(1-\Delta p^{(n)})\cdot[F(t,x)\widehat{u}(t,x) - g(t,x)]\Big) \\
= \widehat{u}_t(x,t) &- (A_L(t) - A_R(t))\,\widehat{u}_x(x,t) - F(x,t)\widehat{u}(x,t) - g(x,t) + o(1) \\
= \widehat{u}_t(t,x) &- A(t)\widehat{u}_x(t,x) - B(t,x)\widehat{u}(t,x) - c(t,x) + o(1) \\
= o(1)&
\end{aligned} \tag{42}$$

The solution $\widehat{u}(t;\cdot)$ to the PDE vanishes outside a compact interval for all $t \in [0,T]$ (Proposition 4) and hence so does $\mathcal{L}^{(n)}(t,\cdot)$. As a result, we also have that

$$\lim_{n\to\infty}\sup_{t\in[0,T]}\|\mathcal{L}^{(n)}(t,\cdot)\|_{L^2} = 0. \tag{43}$$

From (41), one has that $\widehat{u}(t+\Delta t^{(n)},\cdot) = \mathcal{H}_t^{(n)}(\widehat{u}(t,\cdot)) + \Delta t^{(n)}\mathcal{L}^{(n)}(t,\cdot)$. In terms of the error-function

$$\delta\widehat{u}^{(n)}(t,x) := \widehat{u}^{(n)}(t,x) - \widehat{u}(t,x) \tag{44}$$

this yields

$$\begin{aligned}
\delta\widehat{u}^{(n)}(t_{k+1}^{(n)};\cdot) &= \widehat{u}^{(n)}(t_{k+1}^{(n)};\cdot) - \widehat{u}(t_k^{(n)} + \Delta t^{(n)};\cdot) \\
&= \mathcal{H}_k^{(n)}(\widehat{u}^{(n)}(t_k^{(n)},\cdot)) - \mathcal{H}_k^{(n)}(\widehat{u}(t_k^{(n)},\cdot)) - \Delta t^{(n)}\mathcal{L}^{(n)}(t_k^{(n)},\cdot) \\
&= \mathcal{H}_k^{(n)}(\delta\widehat{u}^{(n)}(t_k^{(n)},\cdot)) - \Delta t^{(n)}\mathcal{L}^{(n)}(t_k^{(n)},\cdot),
\end{aligned}$$

due to the linearity of $\mathcal{H}_k^{(n)}$. Using this property iteratively, and putting $(\mathcal{H}^{(n)})^k := \mathcal{H}_1^{(n)}\circ\cdots\circ\mathcal{H}_k^{(n)}$ one finds:

$$\delta\widehat{u}(t_{k+1}^{(n)},\cdot) = (\mathcal{H}^{(n)})^k(\delta\widehat{u}^{(n)}(0,\cdot)) - \Delta t^{(n)}\sum_{i=0}^{k}(\mathcal{H}^{(n)})^{k-i}(\mathcal{L}^{(n)}(t_i^{(n)}),\cdot). \tag{45}$$

From the definition of $\mathcal{H}_t^{(n)}$ together with the fact that the functions F and g are uniformly bounded by assumption, one finds (for large enough $n \in \mathbb{N}$):

$$\begin{aligned}
\|\mathcal{H}_t^{(n)}(v)\|_{L^2} \le\; &\|v\|_{L^2}\left(1 - \Delta p^{(n)}(A_R(t) + A_S(t))\right) + \|T_+^{(n)}(v)\|_{L^2}\cdot\left(\Delta p^{(n)}\cdot A_R(t)\right) \\
&+ \|T_-^{(n)}(v)\|_{L^2}\cdot\left(\Delta p^{(n)}\cdot A_L(t)\right) + \Delta v^{(n)}\left(\|F\|_\infty\|v\|_{L^2} + \|g\|_\infty\right).
\end{aligned}$$

Thus, by the isometry property $\|T_\pm^{(n)}(v)\|_{L^2} = \|v\|_{L^2}$ of the translation operators, there exists a constant $C > 0$ that is independent of $t \in [0,T]$ such that:

$$\sup_{\|v\|_{L^2}=1}\|\mathcal{H}_t^{(n)}(v)\|_{L^2} \le 1 + C\Delta t^{(n)}.$$

In particular, since $k \leq \lfloor \frac{T}{\Delta t^{(n)}} \rfloor$:

$$\begin{aligned}\sup_k \left\| (\mathcal{H}^{(n)})^k (\delta \widehat{u}^{(n)}(0,\cdot)) \right\|_{L^2} &\leq (1 + C\Delta t^{(n)})^{\lfloor T/\Delta t^{(n)} \rfloor} \|\delta \widehat{u}^{(n)}(0,\cdot)\|_{L^2} \\ &\leq e^{CT} \|\delta \widehat{u}^{(n)}(0,\cdot)\|_{L^2} \\ &= o(1),\end{aligned}$$

where the last equality follows from Assumption 1. Similarly, from (43):

$$\sup_{k,i,n} \left\| (\mathcal{H}^{(n)})^{k-i} (\mathcal{L}^{(n)}(t^{(n)}_{i-1}), \cdot) \right\|_{L^2} = o(1).$$

Using the same arguments as before, we conclude that the sum in (45) vanishes uniformly in time. Hence,

$$\lim_{n\to\infty} \sup_{t\in[0,T]} \|\widehat{u}^{(n)}(t,\cdot) - \widehat{u}(t,\cdot)\|_{L^2} = 0.$$

□

Next, we show that the functions $u^{(n)}$ also approximate the PDE. More precisely, the following holds.

Proposition 6. *Under the assumptions of Theorem 1*

$$\lim_{n\to\infty} \sup_{t\in[0,T]} \|\widehat{u}^{(n)}(t;\cdot) - u^{(n)}(t;\cdot)\|_{L^2} = 0.$$

Proof. The proof is similar to that of Proposition 5. By analogy to the operator $\mathcal{H}^{(n)}_t$ we introduce an operator $\widehat{\mathcal{H}}^{(n)}_t$ on L^2 by

$$\begin{aligned}\widehat{\mathcal{H}}^{(n)}_t(v) := v &+ \Delta p^{(n)} \cdot A_R(t) \left[v(\cdot + \Delta x^{(n)}) - v(\cdot) \right] \\ &+ \Delta p^{(n)} \cdot A_L(t) \left[v(\cdot - \Delta x^{(n)}) - v(\cdot) \right] \\ &+ \Delta v^{(n)} \cdot (1 - \Delta p^{(n)}) \left[F^{(n)}(t,\cdot) \cdot v(\cdot) + g^{(n)}(t,\cdot) \right]\end{aligned}$$

where for $t \in [t^{(n)}_k, t^{(n)}_{k+1})$:

$$\begin{aligned}F^{(n)}(t,\cdot) &:= \begin{pmatrix} -f^{(n),C}(\cdot) \cdot p^C(\widehat{\gamma}(t^{(n)}_k)) & 0 \\ 0 & -f^{(n),G}(\cdot) \cdot p^G(\widehat{\gamma}(t^{(n)}_k)) \end{pmatrix} \\ g^{(n)}(t,\cdot) &:= \begin{pmatrix} f^{(n),D}(\cdot) \cdot p^D(\widehat{\gamma}(t^{(n)}_k)) \\ f^{(n),H}(\cdot) \cdot p^H(\widehat{\gamma}(t^{(n)}_k)) \end{pmatrix}.\end{aligned}$$

For the error function $\delta \widehat{u}^{(n)}_k(\cdot) := \widehat{u}^{(n)}_k(\cdot) - u^{(n)}_k(\cdot)$ we then obtain:

$$\begin{aligned}\delta \widehat{u}^{(n)}_{k+1}(\cdot) &= \widehat{\mathcal{H}}^{(n)}_t(\delta \widehat{v}^{(n)}_k(\cdot)) + \Delta v^{(n)} \cdot (1 - \Delta p^{(n)}) \cdot \left(\delta F^{(n)}(\cdot) \cdot \widehat{u}^{(n)}_k(\cdot) + \delta g^{(n)}(\cdot) \right) \\ &=: \widehat{\mathcal{H}}^{(n)}_t(\delta \widehat{u}^{(n)}_k(\cdot)) + \Delta v^{(n)} \cdot (1 - \Delta p^{(n)}) \cdot \widehat{\mathcal{L}}(t^{(n)}_k; \cdot)\end{aligned}$$

where

$$\begin{aligned}\delta F^{(n)}(t,\cdot) &:= \begin{pmatrix} -(f^{(n),C}(\cdot) - f^C(\cdot)) \cdot p^C(\widehat{\gamma}(t^{(n)}_k)) & 0 \\ 0 & -(f^{(n),G}(\cdot) - f^G(\cdot)) \cdot p^G(\widehat{\gamma}(t^{(n)}_k)) \end{pmatrix}, \\ \delta g^{(n)}(t,\cdot) &:= \begin{pmatrix} \left(f^{(n),D}(\cdot) - f^D(\cdot)\right) \cdot p^D(\widehat{\gamma}(t^{(n)}_k)) \\ \left(f^{(n),H}(\cdot) - f^H(\cdot)\right) \cdot p^H(\widehat{\gamma}(t^{(n)}_k)) \end{pmatrix}.\end{aligned}$$

By construction, the grid-point functions $\widehat{u}^{(n)}$ are uniformly bounded. As a result, it follows from Assumption 2 that

$$\lim_{n\to\infty} \sup_{k=0,\dots,\lfloor T/\Delta t^{(n)}\rfloor} \|\widehat{\mathcal{L}}(t_k^{(n)};\cdot)\|_{L^2} = 0.$$

One can now proceed as in the proof of Proposition 5, to conclude that

$$\lim_{n\to\infty} \sup_{t\in[0,T]} \|\widehat{u}^{(n)}(t,\cdot) - u^{(n)}(t;\cdot)\|_{L^2} = 0.$$

□

3.2. Expected volume dynamics and discretized PDEs To show the convergence of the volume density functions we compare the random states $\eta_v^{(n)}$ with the deterministic approximations of the limiting PDE obtained in the previous subsection. For this, we introduce the deterministic step function valued processes $\widetilde{u}^{(n)}$:

$$\widetilde{u}^{(n)}(t,\cdot) := \widetilde{u}_k^{(n)} \quad \text{for } t\in[t_k^{(n}, t_{k+1}^{(n)}) \tag{46}$$

where

$$\begin{cases} \widetilde{u}_{k+1}^{(n)} & := \widetilde{u}_k^{(n)} + \mathbb{E}\left[\mathcal{D}_{v,k}^{(n)}\left(\eta_{\gamma,k}^{(n)}, \widetilde{u}_k^{(n)}\right)\right] \\ \\ \widetilde{u}_0^{(n)} & := v_0^{(n)}. \end{cases} \tag{47}$$

It describes the expected dynamics of the volume density functions for the actual price process; in particular, $\widetilde{u}^{(n)}$ is a stochastic process. By contrast, the process $u^{(n)}$ describes the dynamics of the expected volume density functions when the random evolution of bid and ask prices is replaced by its deterministic limit. We have:

$$\begin{aligned}\|\eta_v^{(n)}(t,\cdot) - \widehat{u}(t,\cdot)\|_{L^2} \le\ & \|\eta_v^{(n)}(t,\cdot) - \widetilde{u}^{(n)}(t,\cdot)\|_{L^2} + \|\widetilde{u}^{(n)}(t,\cdot) - u^{(n)}(t,\cdot)\|_{L^2} \\ & + \|u^{(n)}(t,\cdot) - \widehat{u}^{(n)}(t,\cdot)\| + \|\widehat{u}^{(n)}(t,\cdot) - \widehat{u}(t,\cdot)\|_{L^2}.\end{aligned}$$

The last two terms are deterministic and converges uniformly to zero by Propositions 5 and 6. It remains to show convergence of the first two (random) terms. This will be achieved in the following two subsections.

3.2.1. Estimating the price impact of expected volume dynamics The term $\|\widetilde{u}^{(n)}(t,\cdot) - u^{(n)}(t,\cdot)\|_{L^2}$ measures the impact of the noise in the price process on the expected standing volume. The following proposition shows that it converges to zero in probability, uniformly over compact time intervals.

Proposition 7. *Under the assumptions of Theorem 1 it holds that:*

$$\sup_{t\in[0,T]} \|\widetilde{u}^{(n)}(t,\cdot) - u^{(n)}(t,\cdot)\|_{L^2} \to 0 \quad \textit{in probability as } n\to\infty.$$

Proof. We argue again as in the proof of Proposition 5. Analogously to the operator $\widehat{\mathcal{H}}^{(n)}$ defined in the proof of Proposition 6 we define for $t\in[t_k^{(n)}, t_{k+1}^{(n)})$ the operator

$$\begin{aligned}\widetilde{\mathcal{H}}_k^{(n)}(v) := v &+ \Delta p^{(n)}\cdot \widetilde{A}_R^{(n)}(t)\left[v(\cdot+\Delta x^{(n)}) - v(\cdot)\right] \\ &+ \Delta p^{(n)}\cdot \widetilde{A}_L(t)\left[v(\cdot-\Delta x^{(n)}) - v(\cdot)\right] \\ &+ \Delta v^{(n)}\cdot(1-\Delta p^{(n)})\left[\widetilde{F}^{(n)}(t,\cdot)\cdot v(\cdot) + \widetilde{g}^{(n)}(t,\cdot)\right]\end{aligned}$$

where for $t \in [t_k^{(n)}, t_{k+1}^{(n)})$ we put

$$\widetilde{A}_R^{(n)}(t) := \begin{pmatrix} p^A(\eta_{\gamma,k}^{(n)}) & 0 \\ 0 & p^E(\eta_{\gamma,k}^{(n)}) \end{pmatrix}.$$

The functions $\widetilde{A}_L^{(n)}(t)$, $\widetilde{F}^{(n)}(t,\cdot)$ and $\widetilde{g}^{(n)}(t,\cdot)$ are defined analogously. Let us further put $\delta p_k^{(n),I} := p^I(\hat{\gamma}(t_k^{(n)})) - p^I(\eta_{\gamma,k}^{(n)})$ for $I = \mathbf{A}, \mathbf{B}, \mathbf{E}, \mathbf{F}$ and

$$\delta A_{R,k}^{(n)} := \begin{pmatrix} \delta p_k^{(n),A} & 0 \\ 0 & \delta p_k^{(n),E} \end{pmatrix}, \quad \delta A_{L.k}^{(n)} := \begin{pmatrix} \delta p_k^{(n),B} & 0 \\ 0 & \delta p_k^{(n),F} \end{pmatrix}$$

and denote by $\delta f_k^{(n)}$ and $\delta g_k^{(n)}$ the corresponding quantities for cancelations and limit order placements. Then, the error function $\delta\widetilde{u}_k^{(n)}(\cdot) := \widetilde{u}_k^{(n)}(\cdot) - u_k^{(n)}(\cdot)$ satisfies $\delta\widetilde{u}_0^{(n)} = 0$ and can be represented in terms of the operator $\widetilde{\mathcal{H}}_k^{(n)}$ as follows:

$$\delta\widetilde{u}_{k+1}^{(n)}(\cdot) = \widetilde{\mathcal{H}}_k^{(n)}(\delta\widetilde{u}_k^{(n)}(\cdot)) + \Delta t^{(n)} \cdot \widetilde{\mathcal{L}}_k^{(n)}(t_k^{(n)}, \cdot)$$

where

$$\begin{aligned} \widetilde{\mathcal{L}}_k^{(n)}(t_k^{(n)}, \cdot) := & \frac{\Delta p^{(n)}}{\Delta t^{(n)}} \cdot \delta A_{R,k}^{(n)} \left[u_k^{(n)}(\cdot + \Delta x^{(n)}) - u_k^{(n)}(\cdot) \right] \\ & + \frac{\Delta p^{(n)}}{\Delta t^{(n)}} \cdot \delta A_{L,k}^{(n)} \left[u_k^{(n)}(\cdot - \Delta x^{(n)}) - u_k^{(n)}(\cdot) \right] \\ & + (1 - \Delta p^{(n)}) \cdot \left[\delta f^{(n)} \cdot u_k^{(n)}(\cdot) + \delta g_k^{(n)} \right]. \end{aligned}$$

Corollary 2 establishes

$$\|u_k^{(n)}\|_{L^2} \leq L \quad \text{and} \quad \|T_\pm^{(n)} u_k^{(n)} - u_k^{(n)}\|_{L^2} \leq L \cdot \Delta x^{(n)}$$

for some constant $L < \infty$ that is independent of (n,k). Using our assumptions on the placement, cancelation and event probability functions along with the fact that the functions p^I have bounded gradients and that

$$\lim_{n\to\infty} \sup_{k=0,\dots,\lfloor T/\Delta t^{(n)} \rfloor} |\eta_{\gamma,k}^{(n)} - \widehat{\gamma}(t_k^{(n)})| \to 0 \quad \text{in probability,}$$

this implies

$$\lim_{n\to\infty} \sup_{k=0,\dots,\lfloor T/\Delta t^{(n)} \rfloor} \|\widetilde{\mathcal{L}}_k^{(n)}(t_k^{(n)}, \cdot)\|_{L^2} = 0 \quad \text{in probability.}$$

We can, therefore, argue as in the proof of Proposition 5 to conclude. □

3.2.2. Convergence of volumes to their expected values In this subsection we apply a law of large number for Hilbert space-valued triangular martingale difference arrays (TMDAs) in order to establish the missing convergence to zero of the distance between $\eta_v^{(n)}$ and $\widetilde{u}^{(n)}$. More precisely, our goal is to prove the following result.

PROPOSITION 8. *Suppose the assumption of Theorem 1 hold. Then,*

$$\sup_{t\in[0,T]} \|\eta_v^{(n)}(t,\cdot) - \widetilde{u}^{(n)}(t,\cdot)\|_{L^2} \to 0 \quad \textit{in probability} \quad \textit{as } n\to\infty.$$

Proof. Using the definition of $\eta_{v,k}^{(n)}$ in (35) and $\widetilde{u}_k^{(n)}$ in (46) we see that

$$\widetilde{u}_k^{(n)} = \mathbb{E}\eta_{v,k}^{(n)},$$

conditioned on the price process. As a result,

$$\|\eta_v^{(n)}(t,\cdot) - \widetilde{u}^{(n)}(t,\cdot)\|_{L^2} = \| \sum_{k=0}^{\lfloor t/\Delta t^{(n)} \rfloor} \left(\mathcal{D}_{v,k}^{(n)}(\cdot,\cdot) - \mathbb{E}[\mathcal{D}_{v,k}^{(n)}(\eta_{\gamma,k}^{(n)}, \eta_{v,k}^{(n)})] \right) \|_{L^2}.$$

In order to establish convergence of the sum to zero uniformly in time we introduce the L^2-valued triangular martingale-difference-array

$$Y_k^n := \mathcal{D}_{v,k}^{(n)}(\cdot,\cdot) - \mathbb{E}[\mathcal{D}_{v,k}^{(n)}(\eta_{\gamma,k}^{(n)}, \eta_{v,k}^{(n)})]. \tag{48}$$

If we can show that there exists $\beta > \frac{1}{2}$ such that

$$\sup_{n,k} \left(\frac{1}{\Delta t^{(n)}} \right)^{2\beta} \mathbb{E}[\|Y_k^n\|_{L^2}^2] < \infty, \tag{49}$$

then Theorem 2 and Corollary 1 would guarantee that

$$\lim_{n\to\infty} \mathbb{P}\left[\sup_{0 \le m \le \lfloor T/\Delta t^{(n)} \rfloor} \| \sum_{k=0}^{m} Y_k^n \|_{L^2} > \epsilon \right] = 0, \tag{50}$$

and the proposition would be proved. To establish (49), we need to bound the following terms:

$$\begin{aligned}
&\sup_{n,k} \mathbb{E}\left[\left\| \mathbf{1}_{D_k,H_k} \frac{\Delta v^{(n)}}{\Delta x^{(n)}} M_{v,k}^{(n),D,H} - \mathbb{E}\left[\mathbf{1}_{D_k,H_k} \frac{\Delta v^{(n)}}{\Delta x^{(n)}} M_{v,k}^{(n),D,H} \right] \right\|_{L^2}^2 \right] \\
&\sup_{n,k} \mathbb{E}\left[\left\| \mathbf{1}_{C_k,G_k} \frac{\Delta v^{(n)}}{\Delta x^{(n)}} M_{v,k}^{(n),C,G} \eta_{v,k}^{(n)} - \mathbb{E}\left[\mathbf{1}_{C_k,G_k} \frac{\Delta v^{(n)}}{\Delta x^{(n)}} M_{v,k}^{(n),C,G} \eta_{v,k}^{(n)} \right] \right\|_{L^2}^2 \right] \\
&\sup_{n,k} \mathbb{E}\left[\left\| \mathbf{1}_{A_k,B_k} \left(T_{\pm}^{(n)}\left(\eta_{v_b,k}^{(n)}\right) - \eta_{v_b,k}^{(n)} \right) - \mathbb{E}\left[\mathbf{1}_{A_k,B_k} \left(T_{\pm}^{(n)}\left(\eta_{v_b,k}^{(n)}\right) - \eta_{v_b,k}^{(n)} \right) \right] \right\|_{L^2}^2 \right] \\
&\sup_{n,k} \mathbb{E}\left[\left\| \mathbf{1}_{E_k,F_k} \left(T_{\pm}^{(n)}\left(\eta_{v_s,k}^{(n)}\right) - \eta_{v_s,k}^{(n)} \right) - \mathbb{E}\left[\mathbf{1}_{E_k,F_k} \left(T_{\pm}^{(n)}\left(\eta_{v_s,k}^{(n)}\right) - \eta_{v_s,k}^{(n)} \right) \right] \right\|_{L^2}^2 \right].
\end{aligned}$$

This is done in Lemma 7 in the appendix. In particular, this lemma shows that the placement/cancellation and shift terms are of the order

$$\mathcal{O}\left((\Delta v^{(n)})^2\right) \quad \text{and} \quad \mathcal{O}\left((\Delta v^{(n)})^{2\alpha} + (\Delta v^{(n)})^{2-\alpha}\right),$$

respectively. Since $\alpha \in (1/2, 1)$ the assertion follows for $\beta := \min\{\alpha, 1-\alpha/2\}$. $\square$

4. Application to portfolio liquidation We now discuss how our LOB model could be used to obtain endogenous shape functions for models of optimal portfolio liquidation under market impact. In such models the goal is to find trading strategies that unwind a large number $X > 0$ of shares within a pre-specified time window $[0,T]$ at minimal cost. It is typically assumed that prices are continuous (as in our limiting model), and that the expected distribution of the standing buy (or sell) side volume can be described in terms of a *shape* function

$$f : [0,T] \times \mathbb{R} \to \mathbb{R}_+.$$

The benchmark case studied in the seminal paper of Almgren and Chriss [3] corresponds to a block-shaped order book where $f(t,x) \equiv \delta$, for some $\delta > 0$; discrete-time liquidation problems with more general shape functions have been studied in, e.g. [2]. For a given shape function a sell order of size E_t submitted at time $t \in [0,T]$ moves the best bid price by an amount D_t defined through

$$E_t = \int_0^{D_t} f(t,x)\,dx.$$

Let us denote by $F^{-1}(t,\cdot)$ the inverse of the anti-derivative of the shape function and assume for simplicity that there is no permanent price impact. If the cost of trading is benchmarked against the mid quote, then the cost $c(t,E_t)$ of trading E_t shares is half the spread plus the market impact cost (see [2] and references therein for details):

$$c(t,E_t) = \frac{1}{2}\mathfrak{S}(t)E_t + \int_0^{F^{-1}(t,E_t)} x f(t,x)dx. \tag{51}$$

If orders can be submitted at discrete points in time t_n $(n = 1,...,N)$, the resulting optimization problem is given by:

$$\min_{(E_{t_n})_{n=1}^N} \sum_{i=1}^N c(t_n, E_{t_n}) \quad \text{s.t.} \quad \sum_{n=1}^N E_{t_n} = X. \tag{52}$$

The goal is now to obtain dynamic shape functions from order book data. Empirical placement and cancelation densities $f_i^{(n),I}$ $(i = 0,1,2,...)$ for the visible book, average volumes placed/cancelled and active/passive order arrival rates can be estimated from flow data; examples are given in the next section. Smooth approximations f^I and p^I of the empirical density and probability functions can then be obtained by interpolation. The case of exponential densities is particularly transparent as illustrated by the following example.

Example 4. Assume that we are given empirical Poisson arrival rates $\eta_i^{(n),C}$ and $\eta_i^{(n),D}$ for the price levels $i\cdot\Delta x^{(n)}$ $(i = 0,1,2,...)$ that satisfy

$$\frac{\eta_{i+1}^{(n),C}}{\eta_i^{(n),C}} = e^{-\kappa^C \Delta x^{(n)}} \quad \text{and} \quad \frac{\eta_{i+1}^{(n),D}}{\eta_i^{(n),D}} = e^{-\kappa^D \Delta x^{(n)}}$$

for some constants $\kappa^{C,D} > 0$. Then, the functions $f^{C,D}$ satisfy:

$$f^{C,D}(x) \propto e^{-\kappa^{C,D} x}.$$

Assuming that $\kappa^D > \kappa^C$ (as it is the case for, e.g. Ebay and Facebook; see Table 2 below), the stationary solution is of the form $u_b(t,x) = \kappa_1 e^{-\kappa_2 x}$ for $\kappa_1, \kappa_2 > 0$ and

$$\int_0^\infty u_b(t,x)\,dx = \frac{\kappa_1}{\kappa_2} =: \kappa.$$

Calibrating the shadow book is more difficult. While an approximation of the shadow book should in principle be possible from spread placements, there are several challenges. For instance, one would have to identify orders that genuinely provide liquidity, that is, to clean the data of spread placements that are cancelled after very short periods of time. In particular, one would have to identify and eliminate "ping-orders", i.e. orders sent to detect hidden liquidity in the spread. Subsequently, one would have to estimate average spread placements (as a function of the best bid/ask price or spread). One possibility to bypass this problem is to work with the stationary solution as given in (25). In that case the density functions are only required on the positive half line. This is in fact often done indirectly in portfolio liquidation models when one assumes that the (benchmark) price is a martingale and can hence be treated as a constant in the optimization problem, and the shape function is independent of time. Another possibility is to consider short periods of time over which no price changes occur as in [16]. In any case, assuming that the density functions f^I have been constructed along with the probabilities p^I, the resulting optimization problem reads:

$$\min_{(E_{t_n})} \sum_{n=1}^{N} \left(\frac{1}{2}\mathfrak{S}(t_n)E_{t_n} + \int_0^{U_b^{-1}(t_n,E_{t_n})} x u_b(t_n,x)dx \right) \quad \text{s.t.} \quad \sum_{n=1}^{N} E_{t_n} = X$$

where $U_b^{-1}(t;\cdot)$ denotes the inverse of the anti-derivative of the volume density function $u_b(t;\cdot)$. In the framework of Example 4 it is given by:

$$U_b^{-1}(t,y) = -\frac{1}{\kappa_2}\ln(1-\frac{y}{\kappa}).$$

5. Some calibration results In this section we illustrate how our model parameters, especially the densities f^I and the probabilities p^I can be estimated from flow data. For the reasons outlined above we restrict ourselves to the visible book, i.e. we consider short periods of time over which no price changes occurred. Using LOBSTER[9] data for Jan 2, 2014 we computed empirical buy-side placement and cancellation probabilities in between two consecutive price changes over that day for the first 5 ticks (best bid price and four ticks below the best bid) for Apple (AAPL), Ebay (EBAY), Facebook (FB), Kraft Foods (KRFT) and Microsoft (MSFT).

5.1. Placement and cancellation densities Table 1 reports average numbers of passive orders in between two consecutive price changes, the empirical probabilities $p^{(n),D}$ of a passive buy-side order being a placement, and the empirical placement and cancellation densities $f_i^{(n),D}$ and $f_i^{(n),C}$ for $i = 0,1,2,3,4$ for AAPL, EBAY, FB, KRFT and MSFT. For instance, there were on average 459 passive orders for MFST (corresponding to $\Delta p = 0.002$) and 62 passive orders in between two consecutive active orders for AAPL (corresponding to $\Delta p = 0.016$). The empirical probability of a passive APPL order to be a placement was 0.52; the conditional probability that a placement (at the first five ticks) took place at the top of the book was 0.23. In fact, for APPL the empirical placement and cancellation densities are essentially constant. By contrast, for MSFT almost all of the activity concentrates at the top of the book. For stocks such as MSFT a continuous approximation does not seem appropriate to us.

Stock	Events	Bid 0	Bid 1	Bid 2	Bid 3	Bid 4	Bid 0	Bid 1	Bid 2	Bid 3	Bid 4	$p^{(n),D}$
AAPL	62.0	0.23	0.19	0.2	0.19	0.19	0.21	0.2	0.2	0.20	0.19	0.52
EBAY	33.29	0.56	0.15	0.1	0.1	0.09	0.52	0.2	0.1	0.09	0.09	0.53
FB	21.51	0.51	0.22	0.11	0.09	0.07	0.47	0.28	0.1	0.08	0.07	0.52
MFST	458.98	0.81	0.07	0.04	0.05	0.03	0.79	0.1	0.04	0.04	0.03	0.52
KRFT	16.39	0.32	0.17	0.13	0.16	0.22	0.31	0.21	0.15	0.16	0.17	0.56

Table 1. Empirical buy-side placement (left) and cancellation (right) probabilities between price changes

[9] LOBSTER is an online limit order book data tool, giving access to flow and reconstructed limit order book data for the entire universe of NASDAQ traded stocks. We thank Nikolaus Hautsch for data provision and Gökhan Cebiroglu for assistance with the estimation results.

We approximated $f^{(n),C}$ and $f^{(n),D}$ for EBAY, FB and KRFT using smooth functions $f^{C,D}$. For EBAY and FB exponential densities of the form

$$f^{C,D}(x) = ae^{bx} + c$$

provided very good fits. For KRFT we chose quadratic polynomials:

$$f^{C,D}(x) = ax^2 + bx + c.$$

The fitted parameter values are reported in Table 2. Figure 6 displays the empirical and the theoretical cancellation densities; read bars correspond to empirical densities; blue bars correspond to the values of the integrals of the theoretical (fitted) densities over the respective price bins as in (6) (for $\mathbb{E}[\omega_1^I] = 1$).

	a	b	c
EBAY submission	1.0921	-2.0285	0.092923
EBAY cancellation	0.81613	-1.3455	0.078672
FB submission	0.72488	-1.0779	0.065857
FB cancellation	0.64554	-0.72558	0.025527
KRFT submission	0.033273	-0.19045	0.39952
KRFT cancellation	0.020006	0.13456	0.37005

Table 2. Theoretical placement and cancellation densities

5.2. Volume-weighted placement densities and cancellation ratios Our model assumes that order sizes are random but do not depend on the submission level. Such a dependence can easily be incorporated into the model if we assume that f^D models volume-weighted placement densities. Table 3 reports average buy-side submission and cancellation volumes. Empirical and theoretical volume-weighted placement density functions for KRFT are displayed in Figure 7.

Stock	Bid 0	Bid 1	Bid 2	Bid 3	Bid 4	Bid 0	Bid1	Bid 2	Bid 3	Bid 4
AAPL	167	161	178	190	188	121	123	112	158	179
EBAY	2341	566	455	557	354	2322	566	465	557	354
FB	2122	1894	682	360	240	2236	1435	527	346	207
KRFT	573	420	354	490	558	504	267	176	307	500

Table 3. Average buy-side cancellation (left) and submission (right) volumes.

Estimating cancellation ratios is more difficult. They can only be estimated by either reconstructing the full book from flow data or by tracking each individual order until cancellation or execution. While LOBSTER provides flow data, fully reconstructed books are not readily available. Using NASDAQ ITCH order-message data for the period ranging from January 2011 to April 2011, Cebiroglu and Horst [9] estimated cancellation ratios at the top of the book for a random selection of 31 stocks from the S&P 500. They report cross-sectional top-of-the-book cancellation ratios ranging from 0.16 to 0.28, depending on the stock's liquidity. For APPL, EBAY and MSFT they estimated the cancellation ratios at 0.46, 0.12 and 0.11, respectively. Gao et al. [16] report price-dependent cancellation ratios for the stock Bank of America (BAC). In that paper the authors model the liquidity at a particular price tick as a birth-death process: orders arrive at price-dependent rates relative to the best bid/ask price, and each order is cancelled after an

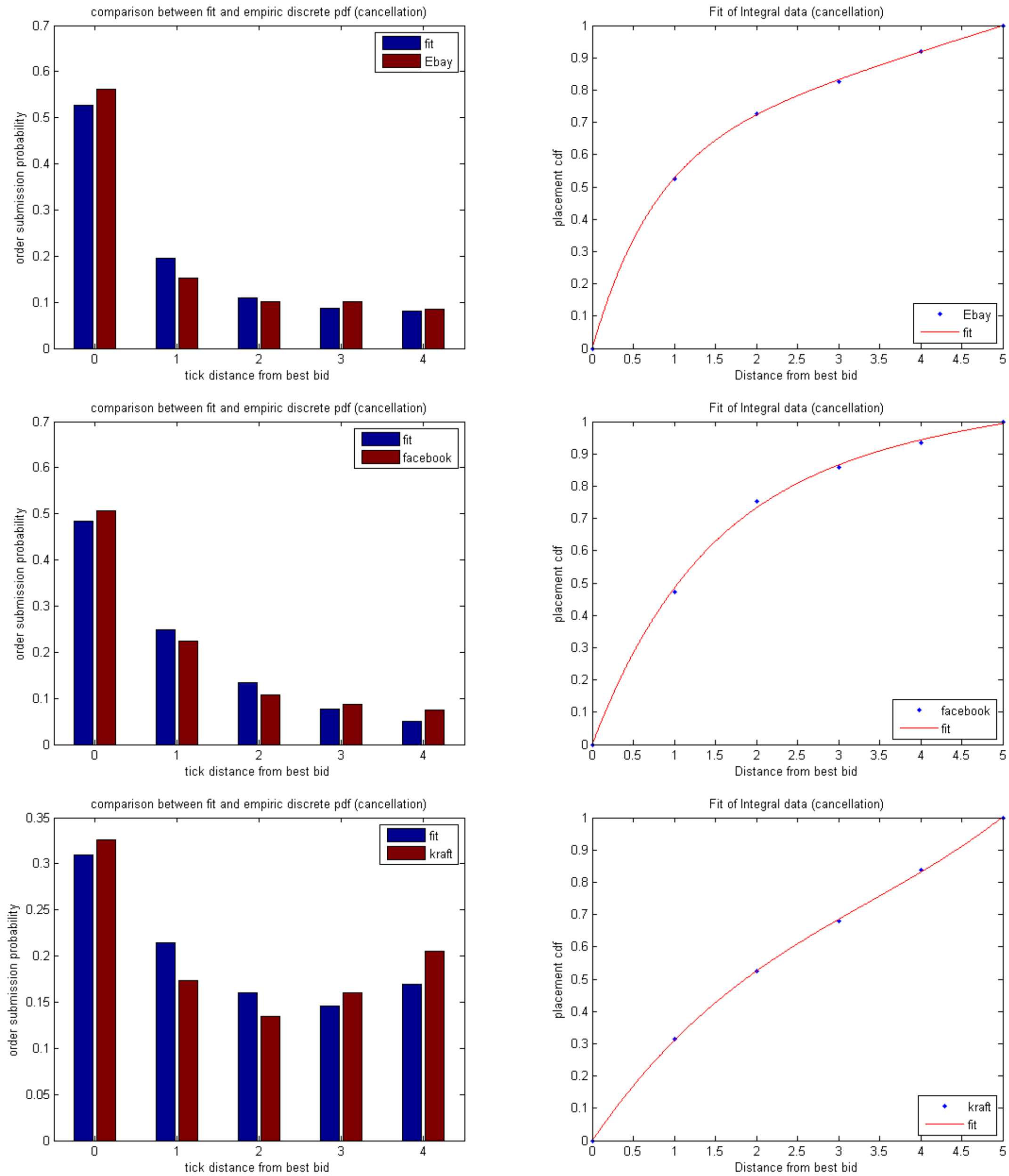

FIGURE 6. Empirical and theoretical cancellation densities for EBAY (top), FB (center), and KRFT (bottom). Left: discrete empirical (red) and theoretical (blue) density; right: distribution function associated with the continuous density f^C

exponentially distributed waiting time. Average cancellation rates per second, $\Theta_A^n(i)$, at the possible ticks $i = 0, 1, 2, ...,$ are estimated by tracking individual orders until cancellation or execution. In distribution, the approach of canceling each order independently after an exponential waiting time is equivalent to proportional cancellation. Hence, the limiting function Θ_A in [16] corresponds to our function $\frac{p^C f^C}{m}$. The calibrations in [16] are based on an empirical analysis of message-level order book data from NYSE Arca in August 2010. For BAC the authors present general summary statistics including the number of buy-side and sell-side events, market order arrival rates, average limit order sizes, and order arrival rates and cancellation ratios for time windows of 5, 10, 15 and

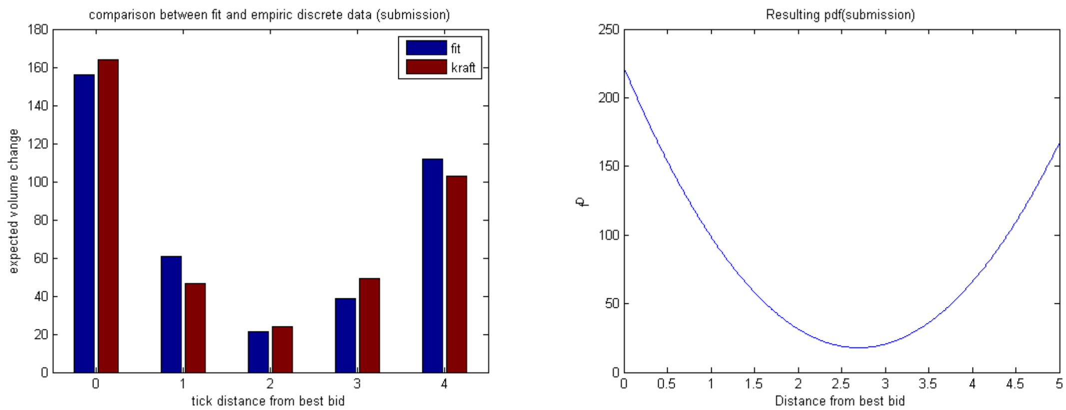

FIGURE 7. Empirical and theoretical volume-weighted placement densities for KRFT; left: discrete empirical (red) and theoretical (blue) volume-weighted density; right: continuous volume-weighted PDF.

20 minutes starting at 12:45pm on August 5, 2010. Their theoretical buy-side cancellation ratios at the first 10 price ticks are given in Table 4.

Bid 0	Bid 1	Bid 2	Bid 3	Bid 4	Bid 5	Bid 6	Bid 7	Bid 8	Bid 9
0.562	0.1	0.151	0.207	0.217	0.11	0	0.004	0.004	0.003

TABLE 4. Empirical cancellation ratios for BAC from [16]

6. Conclusion In this work a law of large numbers for limit order books was established. Starting from order arrival and cancelation rates for all price levels, we showed that the LOB dynamics can be described by a coupled PDE:ODE system when tick and order sizes tend to zero while arrival rates tend to infinity in a particular way. A key insight is that the scaling limit requires two time scales: a fast time scale for passive order arrivals and a comparably slow time scale for active order arrivals. The proof of convergence of volume densities was carried out in three steps: We first showed that the expected LOB dynamics resembles a numerical scheme for hyperbolic PDEs plus noise, provided the random price dynamics is replaced by its deterministic limit. Subsequently, we showed that the impact of the noise in the price process on the volume dynamics vanishes in the limit. Finally, we used a law of large numbers for triangular martingale difference arrays to prove that the LOB model converges to its expected value.

Our model allows for approximation of key order book statistics such as expected price increments, expected standing volumes at future times and expected times to fill. We calibrated placement and cancellation densities to market data for selected stocks and provided numerical simulations that suggest that our model can indeed be used to forecast order book shapes over short periods of time.

Several questions remain open. First, it would be interesting to establish a CLT or, more generally, a diffusion approximation for LOBs. Based on the idea of having different time scales for active and passive order arrivals, Bayer, Horst and Qiu [5] have recently established a first SPDE scaling limit for order books. However, they assume that cancellations are subject to additive (rather than multiplicative) noise so volumes may become negative. Second, it would be interesting to solve models of optimal portfolio liquidation based on our limiting model.

Appendix A: A Law of Large Numbers for Banach-Space-Valued TMDAs In this appendix we prove a law of large numbers for triangular martingale difference arrays taking values in real separable p-uniformly smooth Banach spaces. A Banach space E is called p-uniformly smooth, where $p\in(1,2]$, if

$$\rho_E(\tau)=\sup\left\{\|\frac{x+y}{2}\|+\|\frac{x-y}{2}\|-1:\|x\|=1,\|y\|=\tau\right\}=\mathcal{O}(\tau^p).$$

All Hilbert spaces are 2-uniformly smooth by the parallelogram identity. The spaces C, l^1 and L^1 are not uniformly smooth.

Definition 2. A family of random variables y^n_k, $k=1,...,n$, $n=1,2,...$ defined on some probability space $(\Omega,\mathcal{F},\mathbb{P})$ is called a triangular martingale difference array (TMDA) with respect to a family $\{\mathcal{F}^n\}_{n=1,2,...}$ of filtrations, $\mathcal{F}^n=\{\mathcal{F}^n_k\}^n_{k=0}$, if for all $n=1,2,...$ the sequence $y^n_1,...,y^n_n$ is a $\mathcal{F}^n$-martingale difference sequence (MDS), i.e.

$$\mathbb{E}\left[y^n_k|\mathcal{F}^n_{k-1}\right]=0.$$

If $\{y^n_k\}$ is a TMDA, then for all $n=1,2,...$ one has

$$\mathbb{E}[\sum_{j=1}^k y^n_k|\mathcal{F}^n_{k-1}]=y^n_k,$$

that is, partial sums are martingale. For such martingales, Pisier [26] proved the following moment estimate.

Lemma 3. *Let E be a real separable p-uniformly smooth Banach space $(1\le p\le 2)$. Then, for all $r\ge 1$ there exists a constant $C>0$ such that for all martingales*

$$\left\{\left(\sum_{i=1}^n X_i,\mathcal{G}_n\right)\right\}_{n\ge 1}$$

with values in E, we have

$$\mathbb{E}\left[\sup_{n\ge 1}|X_n|\right]^r\le C\mathbb{E}\left(\sum_{n=1}^\infty|X_i-X_{i-1}|^p\right)^{r/p}.$$

The previous lemma allows us to prove the following weak law of large numbers for triangular martingale difference arrays.

Theorem 2. *Let y^n_k, $k=1,...,n$, $n=1,2,...$ be a TMDA taking values in a real separable p-uniformly smooth Banach space E for $1\le p\le 2$ such that*

$$\sup_{n,k}\mathbb{E}|y^n_k|^p<\infty.$$

Then, for all $\beta>0$ such that $\beta\cdot p>1$ one has for all $\epsilon>0$ that:

$$\lim_{n\to\infty}\mathbb{P}\left[\sup_{1\le m\le n}|\sum_{k=1}^m y^n_k|\ge\epsilon\cdot n^\beta\right]=0.$$

Proof. By Markov's inequality

$$\mathbb{P}\left[\sup_{1\le m\le n}|\sum_{k=1}^{m} y_k^n| \ge \epsilon\cdot n^{\beta}\right] \le \frac{1}{\epsilon n^{q\cdot\beta}}\mathbb{E}\left[\sup_{1\le m\le n}|\sum_{k=1}^{m} y_k^n|\right]^q$$

for all $q\ge 1$. Thus, it follows from Lemma 3 that

$$\begin{aligned}\mathbb{P}\left[\sup_{1\le m\le n}|\sum_{k=1}^{m} y_k^n| \ge \epsilon\cdot n^{\beta}\right] &\le Cn^{-\beta\cdot q}\mathbb{E}\left[\sum_{k=1}^{n}|y_k^n|^p\right]^{q/p}\\ &\le Cn^{-\beta\cdot q+\frac{q}{p}}\end{aligned}$$

for a generic constant $C>0$ since the random variables y_k^n have a uniformly bounded p:th moment. Hence, the assertion follows as soon as $-\beta\cdot q+\frac{q}{p}<0$. This holds for all $q>0$ as $\beta\cdot p>1$. □

As an immediate corollary from the preceding theorem one obtains the following law of large numbers for TMDAs.

Corollary 1. *Let y_k^n, $k=1,\dots,n$, $n=1,2,\dots$ be a TMDA taking values in a real separable 2-uniformly smooth Banach space E such that*

$$\sup_{n,k}\left(n^{2\beta}\mathbb{E}|y_k^n|^2\right)<\infty$$

for some $\beta>\frac{1}{2}$. Then,

$$\lim_{n\to\infty}\sup_{1\le m\le n}|\sum_{k=1}^{m} y_k^n| = 0 \quad \textit{in probability.}$$

Proof. We apply Theorem 2 to the TMDA

$$\hat{y}_k^n := n^{\beta} y_k^n.$$

Then

$$\lim_{n\to\infty}\mathbb{P}\left[\sup_{1\le m\le n}|\sum_{k=1}^{m}\hat{y}_k^n|\ge\epsilon\cdot n^{\beta}\right]=0.$$

Hence the assertion follows from:

$$\mathbb{P}\left[\sup_{1\le m\le n}|\sum_{k=1}^{m}\hat{y}_k^n|\ge\epsilon\cdot n^{\beta}\right]=\mathbb{P}\left[\sup_{1\le m\le n}|\sum_{k=1}^{m} y_k^n|\ge\epsilon\right]=0.$$

□

Appendix B: Properties of volume density functions In this appendix we prove some properties of the volume density functions. In particular, we show that the sequences $\{\eta_{v,k}^{(n)}\}$ take values in L^2 almost surely. We first use an induction argument to establish a useful representation of the volume density function.

Lemma 4. *The buy side volume density function $\eta_{v_b}^{(n)}$ satisfies:*

$$\begin{aligned}\eta_{v_b,k}^{(n)} = &\left(\left(T_+^{(n)}\right)^{\sum_{i=0}^{k-1}\mathbf{1}_i^{(n),A}}\circ\left(T_-^{(n)}\right)^{\sum_{i=0}^{k-1}\mathbf{1}_i^{(n),B}}\right)\left(v_{b,0}^{(n)}\right)\\ &+\frac{\Delta v^{(n)}}{\Delta x^{(n)}}\cdot\sum_{i=0}^{k-1}\left\{\left(\left(T_+^{(n)}\right)^{\sum_{j=i+1}^{k-1}\mathbf{1}_j^{(n),A}}\circ\left(T_-^{(n)}\right)^{\sum_{j=i+1}^{k-1}\mathbf{1}_j^{(n),B}}\right)\left(-M_{v,i}^{(n),C}\eta_{v_b,i}^{(n)}\mathbf{1}_i^{(n),C}+M_{v,i}^{(n),D}\mathbf{1}_i^{(n),D}\right)\right\}.\end{aligned}$$

Proof. For $k=0$, the equation holds by definition. Let us therefore assume it holds for all $k \le p$. For $k=p+1$ one then obtains

$$\begin{aligned}
\eta^{(n)}_{v_b,p+1} &= \eta^{(n)}_{v_b,p} + \left(T^{(n)}_{+}(\eta^{(n)}_{v_b,p}) - \eta^{(n)}_{v_b,p}\right)\mathbf{1}^{(n),A}_{p} + \left(T^{(n)}_{-}(\eta^{(n)}_{v_b,p}) - \eta^{(n)}_{v_b,p}\right)\mathbf{1}^{(n),B}_{p} \\
&\qquad - \frac{\Delta v^{(n)}}{\Delta x^{(n)}} \cdot M^{(n),C}_{v,p}\eta^{(n)}_{v_b,p}\mathbf{1}^{(n),C}_{p} + \frac{\Delta v^{(n)}}{\Delta x^{(n)}} \cdot M^{(n),D}_{v,p}\mathbf{1}^{(n),D}_{p} \\
&= \left(\left(T^{(n)}_{+}\right)^{\mathbf{1}^{(n),A}_{p}} \circ \left(T^{(n)}_{-}\right)^{\mathbf{1}^{B}_{p}}\right)\left(\eta^{(n)}_{v_b,p}\right) - M^{(n),C}_{p}\eta^{(n)}_{v_b,p}\mathbf{1}^{(n),C}_{p} + M^{(n),D}_{p}\mathbf{1}^{(n),D}_{p} \\
&= \left(\left(T^{(n)}_{+}\right)^{\sum_{i=0}^{p}\mathbf{1}^{(n),A}_{i}} \circ \left(T^{(n)}_{-}\right)^{\sum_{i=0}^{p}\mathbf{1}^{(n),B}_{i}}\right)\left(v^{(n)}_{b,0}\right) \\
&\quad + \frac{\Delta v^{(n)}}{\Delta x^{(n)}} \cdot \sum_{i=0}^{p}\left\{\left(\left(T^{(n)}_{+}\right)^{\sum_{j=i+1}^{p}\mathbf{1}^{(n),A}_{j}} \circ \left(T^{(n)}_{-}\right)^{\sum_{j=i+1}^{p}\mathbf{1}^{(n),B}_{j}}\right)\left(-M^{(n),C}_{v,i}\eta^{(n)}_{v_b,i}\mathbf{1}^{(n),C}_{i} + M^{(n),D}_{v,i}\mathbf{1}^{(n),D}_{i}\right)\right\}.
\end{aligned}$$

□

B.1. Boundedness of volume densities Using the isometry property of the translation operator we deduce that the L^2 norm of the volume density function can be estimated from above by considering a model with only passive order placements. In a similar way we can estimate the expected order book hight at any given price tick. More precisely, we have the following result.

LEMMA 5. *The expected L^2-norm of the volume density function is uniformly bounded:*

$$\sup_{n\in\mathbb{N},\, k=0,\dots,\lfloor T/\Delta t^{(n)}\rfloor} \mathbb{E}\|\eta^{(n)}_{v_b,k}\|^2_{L^2} \le C \tag{53}$$

for some constant $C<\infty$. Likewise, the expected order book hight is uniformly bounded, i.e. if we put $\eta^{(n)}_{v_b,k} = (\eta^{(n),j}_{v_b,k})_{j\in\mathbb{Z}}$, then :

$$\sup_{j\in\mathbb{Z},\, n\in\mathbb{N},\, k=0,\dots,\lfloor T/\Delta t^{(n)}\rfloor} \mathbb{E}|\eta^{(n),j}_{v_b,k}|^2 \le C. \tag{54}$$

Proof. It is enough to consider a model with only order placements where $\omega^D_k = 1$ a.s. W.l.o.g. we may also assume that $|\pi^D_k| \le 1$ a.s. and $\eta^{(n)}_{v_b,0} \equiv 0$. Furthermore we may as well use a representation of the volume densities in *absolute* rather than *relative* coordinates. In such a model, $\mathbb{E}\|\eta^{(n)}_{v_b,k}\|^2_{L^2}$ is of the form

$$\mathbb{E}\|\eta^{(n)}_{v_b,k}\|^2_{L^2} = \left(\frac{\Delta v^{(n)}}{\Delta x^{(n)}}\right)^2 \mathbb{E}\sum_{j\in\mathbb{Z}}\left(\sum_{i=1}^{k} a^{(n)}_{i,j}\right)^2 \cdot \Delta x^{(n)}$$

where $a^{(n)}_{i,j} := \mathbf{1}_{\{\pi^D_i \in [x^{(n)}_j, x^{(n)}_{j+1})\}}$ with the distribution of the random variables π^D_i properly adjusted to account for the representation in absolute coordinates. Since the random variables π^D_i have compact support, only finitely many summands are non-zero and we may rearrange terms. Using conditional independence of the placement variables though time, this yields:

$$\mathbb{E}\|\eta^{(n)}_{v_b,k}\|^2_{L^2} = \frac{\left(\Delta v^{(n)}\right)^2}{\Delta x^{(n)}} \sum_{i=1}^{k}\sum_{j\in\mathbb{Z}} \mathbb{E}\left(a^{(n)}_{i,j}\right)^2 + \frac{\left(\Delta v^{(n)}\right)^2}{\Delta x^{(n)}} \sum_{i,i'=1, i\neq i'}^{k}\sum_{j\in\mathbb{Z}} \mathbb{E}a^{(n)}_{i,j}\mathbb{E}a^{(n)}_{i',j}.$$

Using the fact that no placements take place at price levels with a distance of more than 1 from the prevailing best bid/ask price:

$$\begin{aligned}
\mathbb{E}\left(a^{(n)}_{i,j}\right)^2 &\le \|f^D\|_\infty \Delta x^{(n)} \mathbf{1}_{\{|j-\eta^{(n)}_{\gamma,i}|\le 1\}} \\
\left|\mathbb{E}a^{(n)}_{i,j}\mathbb{E}a^{(n)}_{i',j}\right| &\le \|f^D\|^2_\infty \left(\Delta x^{(n)}\right)^2 \mathbf{1}_{\{|j-\eta^{(n)}_{\gamma,i}|\le 1\}}\mathbf{1}_{\{|j-\eta^{(n)}_{\gamma,i'}|\le 1\}}.
\end{aligned}$$

In particular, the inner sums extend over at most $\frac{2}{\Delta x^{(n)}}+1$ terms. As a result, our scaling assumptions guarantee that

$$\sup_{n,k=1,\dots,\lfloor T/\Delta t^{(n)}\rfloor} \mathbb{E}\|\eta^{(n)}_{v_b,k}\|^2_{L^2}<\infty.$$

The second assertion follows analogously as

$$\mathbb{E}|\eta^{(n),j}_{v_b,k}|^2=\frac{\left(\Delta v^{(n)}\right)^2}{\Delta x^{(n)}}\mathbb{E}\left(\sum_{i=1}^k a^{(n)}_{i,j}\right)^2.$$

□

B.2. Norm estimates The next result will be used to prove a Lipschitz continuity property of the grid-point approximation of the limiting PDE.

LEMMA 6. *There exists a constant $C<0$ such that for all $n\in\mathbb{N}$ and $k=0,\dots,\lfloor T/\Delta t^{(n)}\rfloor$:*

$$\left\|\mathbb{E}\left[T^{(n)}_{\pm}(\eta^{(n)}_{v_b,k})-\eta^{(n)}_{v_b,k}\right]\right\|_{L^2}\le C\cdot\Delta x^{(n)}.$$

Proof. Using Lemma 4 and the linearity of the translation operator $T^{(n)}_+$ it follows that a.s.

$$\begin{aligned}
&T^{(n)}_+\left(\eta^{(n)}_{v_b,k}\right)-\eta^{(n)}_{v_b,k}\\
&=\left(\left(T^{(n)}_+\right)^{\sum_{i=0}^{k-1}\mathbf{1}^A_i}\circ\left(T^{(n)}_-\right)^{\sum_{i=0}^{k-1}\mathbf{1}^B_i}\right)\left(T^{(n)}_+\left(v^{(n)}_{b,0}\right)-v^{(n)}_{b,0}\right) &(55)\\
&\quad+\frac{\Delta v^{(n)}}{\Delta x^{(n)}}\cdot\sum_{i=0}^{k-1}\left(\left(T^{(n)}_+\right)^{\sum_{j=i+1}^{k-1}\mathbf{1}^A_j}\circ\left(T^{(n)}_-\right)^{\sum_{j=i+1}^{k-1}\mathbf{1}^B_j}\right)\left(\left[T^{(n)}_+\left(M^{(n),D}_i\right)-M^{(n),D}_i\right]\mathbf{1}^D_i\right) &(56)\\
&\quad-\frac{\Delta v^{(n)}}{\Delta x^{(n)}}\cdot\sum_{i=0}^{k-1}\left(\left(T^{(n)}_+\right)^{\sum_{j=i+1}^{k-1}\mathbf{1}^A_j}\circ\left(T^{(n)}_-\right)^{\sum_{j=i+1}^{k-1}\mathbf{1}^B_j}\right)\left(\left[T^{(n)}_+\left(M^{(n),C}_i\eta^{(n)}_{v_b,i}\right)-M^{(n),C}_i\eta^{(n)}_{v_b,i}\right]\mathbf{1}^C_i\right) &(57)
\end{aligned}$$

Taking the expected value and norms in (57) we find:

$$\begin{aligned}
&\left\|\mathbb{E}\left[T^{(n)}_+\left(\eta^{(n)}_{v_b,k}\right)-\eta^{(n)}_{v_b,k}\right]\right\|_{L^2}\\
&\le\left\|T^{(n)}_+\left(v^{(n)}_{b,0}\right)-v^{(n)}_{b,0}\right\|_{L^2}+\frac{\Delta v^{(n)}}{\Delta x^{(n)}}\sum_{i=0}^{k-1}\left\|T^{(n)}_+\left(\mathbb{E}\left[M^{(n),D}_i\right]\right)-\mathbb{E}\left[M^{(n),D}_i\right]\right\|_{L^2}.\\
&\qquad+\frac{\Delta v^{(n)}}{\Delta x^{(n)}}\sum_{i=0}^{k-1}\left\|T^{(n)}_+\left(\mathbb{E}\left[M^{(n),C}_{v,i}\eta^{(n)}_{v_b,i}\right]\right)-\mathbb{E}\left[M^{(n),C}_{v,i}\eta^{(n)}_{v_b,i}\right]\right\|_{L^2}. &(58)
\end{aligned}$$

By Assumptions 1 and 3 there exists a constant $K<\infty$ such that $\left\|T^{(n)}_+\left(v^{(n)}_{b,0}\right)-v^{(n)}_{b,0}\right\|_{L^2}\le K\Delta x^{(n)}$ and

$$\begin{aligned}
\frac{1}{\Delta x^{(n)}}\left\|T^{(n)}_+\left(\mathbb{E}\left[M^{(n),D}_i\right]\right)-\mathbb{E}\left[M^{(n),D}_i\right]\right\|_{L^2}&=\left\|T^{(n)}_+\left(f^{(n),D}\right)-f^{(n),D}\right\|_{L^2}\\
&\le K\Delta x^{(n)}.
\end{aligned}$$

As for the cancellation terms, independence of the event dynamics from the standing volumes yield:

$$\begin{aligned}
\frac{1}{\Delta x^{(n)}}T^{(n)}_+\left(\mathbb{E}\left[M^{(n),C}_i\eta^{(n)}_{v_b,i}\right]\right)&=T^{(n)}_+\left(f^{(n),C}\mathbb{E}\eta^{(n)}_{v_b,i}\right),\\
\frac{1}{\Delta x^{(n)}}\mathbb{E}\left[M^{(n),C}_i\eta^{(n)}_{v_b,i}\right]&=f^{(n),C}\mathbb{E}\eta^{(n)}_{v_b,i}.
\end{aligned}$$

In view of the second assertion of Lemma 5 and using the fact that $f^{(n),C}$ is bounded along with Assumption 3 we find a constant $K<\infty$ such that:

$$\begin{aligned}
&\frac{1}{\Delta x^{(n)}}\left\|T_+^{(n)}\left(\mathbb{E}\left[M_i^{(n),C}\eta_{v_b,i}^{(n)}\right]\right)-\mathbb{E}\left[M_i^{(n),C}\eta_{v_b,i}^{(n)}\right]\right\|_{L^2}\\
\leq&\left\|T_+^{(n)}\left(f^{(n),C}\right)\mathbb{E}\left[T_+^{(n)}\left(\eta_{v_b,i}^{(n)}\right)-\eta_{v_b,i}^{(n)}\right]\right\|_{L^2}\\
&+\left\|\left(T^{(n)}\left(f^{(n),C}\right)-f^{(n),C}\right)\mathbb{E}\eta_{v_b,i}^{(n)}\right\|_{L^2}\\
\leq& K\left(\mathbb{E}\|T_+^{(n)}\left(\eta_{v_b,i}^{(n)}\right)-\eta_{v_b,i}^{(n)}\|_{L^2}+\Delta x^{(n)}\right).
\end{aligned}$$

Altogether, we arrive at the following estimate:

$$\left\|\mathbb{E}\left[T_+^{(n)}\left(\eta_{v_b,k}^{(n)}\right)-\eta_{v_b,k}^{(n)}\right]\right\|_{L^2}\leq K\Delta x^{(n)}+K\Delta v^{(n)}\sum_{i=0}^{\lfloor T/\Delta t^{(n)}\rfloor}\left\|\mathbb{E}\left[T_+^{(n)}\left(\eta_{v_b,i}^{(n)}\right)-\eta_{v_b,i}^{(n)}\right]\right\|_{L^2}. \tag{59}$$

Hence, it follows from Gronwall's lemma that

$$\sup_{k=0,\dots,T/\Delta t^{(n)}}\left\|\mathbb{E}\left[T_+^{(n)}\left(\eta_{v_b,k}^{(n)}\right)-\eta_{v_b,k}^{(n)}\right]\right\|_{L^2}=\mathcal{O}\left(\Delta x^{(n)}\right).$$

□

COROLLARY 2. *There exists a constant $C>0$ such that*

$$\|u^{(n)}(t,\cdot+\Delta x^{(n)})-u^{(n)}(t,\cdot)\|_{L^2}\leq C\cdot\Delta x^{(n)}.$$

Moreover,

$$\sup_{n\in\mathbb{N},t\in[0,T]}\|u^{(n)}(t;\cdot)\|_{L^2}<\infty.$$

Proof. In order to establish the first assertion we represent the functions $u^{(n)}$ as

$$u_k^{(n)}=\mathbb{E}\zeta_k^{(n)} \tag{60}$$

where

$$\begin{cases}\zeta_{k+1}^{(n)} & :=\zeta_k^{(n)}+\mathcal{D}_{v,k}^{(n)}\left(\gamma(t_k^{(n)}),\zeta_k^{(n)}\right)\\ \zeta_0^{(n),j} & :=v_0^{(n),j}\end{cases}. \tag{61}$$

For $t\in[k\cdot\Delta t^{(n)},(k+1)\cdot\Delta^{(n)})$ the preceding lemma then implies:

$$\|u^{(n)}(t,\cdot\pm\Delta x^{(n)})-u^{(n)}(t,\cdot)\|_{L^2}=\left\|\mathbb{E}\left[T_\pm^{(n)}\left(\zeta_k^{(n)}\right)-\zeta_k^{(n)}\right]\right\|_{L^2}\leq C\Delta x^{(n)}.$$

The second assertion follows from (60) together with Lemma 5:

$$\|u^{(n)}(t_k^{(n)};\cdot)\|_{L^2}=\|\mathbb{E}\zeta_k^{(n)}\|_{L^2}\leq\mathbb{E}\|\zeta_k^{(n)}\|_{L^2}\leq C.$$

□

Using Lipschitz continuity of f^I along with the point wise shift estimate of the initial volume densities of Assumption 1 the following result can be established by analogy to Corollary 6.

COROLLARY 3.

$$\sup_{n\in\mathbb{N},\, j\in\mathbb{Z},\, k=0,\dots,\lfloor T/\Delta t^{(n)}\rfloor} \left|\mathbb{E}\left[\eta^{(n),j+1}_{v_b,k} - \eta^{(n),j}_{v_b,k}\right]\right| = \mathcal{O}(\Delta x^{(n)}). \tag{62}$$

We close this appendix with norm estimates which are key to the proof of Proposition 8.

LEMMA 7. *The following norm estimates hold:*

$$\sup_{n,k}\mathbb{E}\left[\left\|\mathbf{1}_{D_k,H_k}\frac{\Delta v^{(n)}}{\Delta x^{(n)}}M^{(n),D,H}_{v,k} - \mathbb{E}\left[\mathbf{1}_{D_k,H_k}\frac{\Delta v^{(n)}}{\Delta x^{(n)}}M^{(n),D,H}_{v,k}\right]\right\|^2_{L^2}\right] = \mathcal{O}\left((\Delta v^{(n)})^2\right)$$

$$\sup_{n,k}\mathbb{E}\left[\left\|\mathbf{1}_{C_k,G_k}\frac{\Delta v^{(n)}}{\Delta x^{(n)}}M^{(n),C,G}_{v,k}\eta^{(n)}_{v,k} - \mathbb{E}\left[\mathbf{1}_{C_k,G_k}\frac{\Delta v^{(n)}}{\Delta x^{(n)}}M^{(n),C,G}_{v,k}\eta^{(n)}_{v,k}\right]\right\|^2_{L^2}\right] = \mathcal{O}\left((\Delta v^{(n)})^2\right)$$

$$\sup_{n,k}\mathbb{E}\left[\left\|\mathbf{1}_{A_k,B_k}\left(T^{(n)}_{\pm}\left(\eta^{(n)}_{v_b,k}\right) - \eta^{(n)}_{v_b,k}\right) - \mathbb{E}\left[\mathbf{1}_{A_k,B_k}\left(T^{(n)}_{\pm}\left(\eta^{(n)}_{v_b,k}\right) - \eta^{(n)}_{v_b,k}\right)\right]\right\|^2_{L^2}\right] = \mathcal{O}\left((\Delta v^{(n)})^{2\alpha} + (\Delta v^{(n)})^{2-\alpha}\right)$$

$$\sup_{n,k}\mathbb{E}\left[\left\|\mathbf{1}_{E_k,F_k}\left(T^{(n)}_{\pm}\left(\eta^{(n)}_{v_s,k}\right) - \eta^{(n)}_{v_s,k}\right) - \mathbb{E}\left[\mathbf{1}_{E_k,F_k}\left(T^{(n)}_{\pm}\left(\eta^{(n)}_{v_s,k}\right) - \eta^{(n)}_{v_s,k}\right)\right]\right\|^2_{L^2}\right] = \mathcal{O}\left((\Delta v^{(n)})^{2\alpha} + (\Delta v^{(n)})^{2-\alpha}\right).$$

Proof. The first two estimates follow from boundedness of the density functions f^I along with independence of the event dynamics from volumes and (53). In order to establish the third and fourth estimate we have to prove that

$$\sup_{n,k}\left\{\Delta p^{(n)}\cdot\mathbb{E}\left[\|T^{(n)}_{+}\left(\eta^{(n)}_{v_b,k}\right) - \eta^{(n)}_{v_b,k}\|^2_{L^2}\right]\right\} = \mathcal{O}\left((\Delta v^{(n)})^{2\alpha} + (\Delta v^{(n)})^{2-\alpha}\right).$$

To this end we use a representation of $T^{(n)}_{+}\eta^{(n)}_{v_b,k} - \eta^{(n)}_{v_b,k}$ as in Lemma 6 but in *absolute* rather than *relative* coordinates. This means that the shift terms drop out of the representation but the probabilities of placements and cancellations need to be properly adjusted.

Assumption 1 allows us to bound the impact of the initial condition (55) by a term of the order $(\Delta x^{(n)})^2$. To compute the norm of the sum in (56) we need to compute a term of the form

$$\left(\frac{\Delta v^{(n)}}{\Delta x^{(n)}}\right)^2\mathbb{E}\sum_{j\in\mathbb{Z}}\left(\sum_{i=1}^{k}a^{(n)}_{i,j}\right)^2\cdot\Delta x^{(n)}$$

where

$$a^{(n)}_{i,j} = \begin{cases} 1 & \text{if } \pi^D_i \in [(j-1)\cdot\Delta x^{(n)}, j\cdot\Delta x^{(n)}) \\ -1 & \text{if } \pi^D_i \in [j\cdot\Delta x^{(n)}, (j+1)\cdot\Delta x^{(n)}) \\ 0 & \text{else} \end{cases}.$$

In particular there exists a constant $K<\infty$ such that

$$\mathbb{E}\left(a^{(n)}_{i,j}\right)^2 \le K\Delta x^{(n)}\mathbf{1}_{\{|j-\eta^{(n)}_{\gamma,i}|\le 1\}}$$
$$\left|\mathbb{E}a^{(n)}_{i,j}\right| \le K\left(\Delta x^{(n)}\right)^2\mathbf{1}_{\{|j-\eta^{(n)}_{\gamma,i}|\le 1\}}$$

where the second inequality follows from (5); the indicator functions account for the representation of volumes in *absolute* coordinates. Using the fact that the random variables π^D_k have compact support and that events are conditionally independent through time, we can now argue as in the proof of Lemma 5 to deduce that:

$$\Delta p^{(n)}\left(\frac{\Delta v^{(n)}}{\Delta x^{(n)}}\right)^2\mathbb{E}\sum_{j\in\mathbb{Z}}\left(\sum_{i=1}^{k}a^{(n)}_{i,j}\right)^2\cdot\Delta x^{(n)} \le K\Delta p^{(n)}\frac{(\Delta v^{(n)})^2}{\Delta x^{(n)}}\left(\frac{1}{\Delta v^{(n)}} + \frac{1}{(\Delta v^{(n)})^2}\frac{1}{\Delta x^{(n)}}(\Delta x^{(n)})^4\right)$$
$$= K\left((\Delta v^{(n)})^{2\alpha} + (\Delta v^{(n)})^{2-\alpha}\right).$$

To compute the norm of the sum in (57) we need to compute a similar term, but with $a_{i,j}^{(n)}$ replaced by

$$b_{i,j}^{(n)} = \begin{cases} \eta_{v_b,k}^{(n),j} & \text{if } \pi_i^D \in [(j-1)\cdot\Delta x^{(n)}, j\cdot\Delta x^{(n)}) \\ -\eta_{v_b,k}^{(n),j-1} & \text{if } \pi_i^D \in [j\cdot\Delta x^{(n)}, (j+1)\cdot\Delta x^{(n)}) \\ 0 & \text{else} \end{cases}.$$

Using (54) we have again that $\mathbb{E}\left(a_{i,j}^{(n)}\right)^2 \leq K\Delta x^{(n)}\mathbf{1}_{\{|j-\eta_{\gamma,i}^{(n)}|\leq 1\}}$. Using independence of the event dynamics from volumes along with Lipschitz continuity of f^C we also obtain a constant $K<\infty$ such that:

$$\begin{aligned}\left|\mathbb{E}a_{i,j}^{(n)}\right| &\leq \left|\mathbb{P}[\pi_i^D \in [j\cdot\Delta x^{(n)}, (j+1)\cdot\Delta x^{(n)})]\cdot\mathbb{E}\eta_{v_b,i}^{(n),j} - \mathbb{P}[\pi_i^D \in [(j-1)\cdot\Delta x^{(n)}, j\cdot\Delta x^{(n)})]\cdot\mathbb{E}\eta_{v_b,i}^{(n),j-1}\right| \\ &\leq K\left\{\left(\Delta x^{(n)}\right)^2 + |\mathbb{E}\eta_{v_b,i}^{(n),j} - \mathbb{E}\eta_{v_b,i}^{(n),j-1}|\cdot\mathbb{P}[\pi_i^D \in [(j-1)\cdot\Delta x^{(n)}, j\cdot\Delta x^{(n)})]\right\}.\end{aligned}$$

Hence it follows from Corollary 3 that

$$\left|\mathbb{E}a_{i,j}^{(n)}\right| \leq K\left(\Delta x^{(n)}\right)^2$$

and the assertion follows as in the case of placements.

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