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Ulrich Horst [a] & Felix Naujokat [a]

[a] Institut füür Mathematik, Humboldt-Universitäät Berlin, Unter den Linden 6, D-10099 Berlin, Germany

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On derivatives with illiquid underlying and market manipulation

ULRICH HORST and FELIX NAUJOKAT*

Institut für Mathematik, Humboldt-Universität Berlin, Unter den Linden 6, D-10099 Berlin, Germany

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In illiquid markets, option traders may have an incentive to increase their portfolio value by using their impact on the dynamics of the underlying. We provide a mathematical framework to construct optimal trading strategies under market impact in a multi-player framework by introducing strategic interactions into the model of Almgren [*Appl. Math. Finance*, 2003, **10**(1), 1–18]. Specifically, we consider a financial market model with several strategically interacting players who hold European contingent claims and whose trading decisions have an impact on the price evolution of the underlying. We establish the existence and uniqueness of equilibrium results for risk-neutral and CARA investors and show that the equilibrium dynamics can be characterized in terms of a coupled system of possibly nonlinear PDEs. For the linear cost function used by Almgren, we obtain a (semi) closed-form solution. Analysing this solution, we show how market manipulation can be reduced.

Keywords: Liquidity modelling; Derivatives pricing; Stochastic models; Game theory; Market manipulation

1. Introduction

Standard financial market models assume that asset prices follow an exogenous stochastic process and that all transactions can be settled at the prevailing price without any impact on market dynamics. The assumption that all trades can be carried out at exogenously given prices is appropriate for small investors who trade only a negligible proportion of the overall daily trading volume; it is not appropriate for institutional investors trading large blocks of shares over a short time span. Trading large amounts of shares is likely to move stock prices in an unfavorable direction. This is a clear disadvantage for traders who need to liquidate or acquire large portfolios. In derivative markets the situation is more ambiguous. A trader who holds a large number of options may have an incentive to utilize his impact on the dynamics of the underlying and to move the option value in a favorable direction if the increase in the option value outweighs the trading costs in the underlying.† Kumar and Seppi (1992) call such trading behavior 'punching the close'. The present paper provides a continuous-time framework to model the interaction between several investors who have an incentive to punch the close. We set this up as a stochastic differential game and establish the existence and uniqueness of Markov equilibria for risk-neutral and CARA investors. For certain cases we have explicit solutions that allow us to discuss some ideas concerning how manipulation in the sense of 'punching the close' could potentially be reduced.

Our work builds on previous research in at least three different fields. The first is the mathematical modeling of illiquid financial markets. The role of liquidity as a source of financial risk has been extensively investigated in both the mathematical finance and financial economics literature over the last couple of years. Much of the literature focusses on either optimal hedging and portfolio liquidation strategies for a single large investor under market impact (Çetin *et al*. 2004, Alfonsi *et al*. 2010, Rogers and Singh 2010), predatory trading (Brunnermeier and Pedersen 2005, Carlin *et al*. 2007, Schied and Schöneborn 2007) or the role of derivative securities, including the problem of market manipulation using options (Kumar and Seppi 1992, Jarrow 1994). It has been shown by Jarrow (1994), for instance, that by introducing derivatives into an otherwise complete and

*Corresponding author. Email: naujokat@mathematik.hu-berlin.de

†Gallmeyer and Seppi (2000) provide some evidence that, in illiquid markets, option traders are in fact able to increase a derivative's value by moving the price of the underlying.

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arbitrage-free market, certain manipulation strategies for a large trader may appear, such as market corners and front runs. Schönbucher and Wilmott (2000) discuss an illiquid market model where a large trader can influence the stock price with vanishing costs and risk. They argue that the risk of manipulation on the part of the large trader makes the small traders unwilling to trade derivatives any longer. In particular, they predict that the option market breaks down. Our analysis indicates that markets do not necessarily break down when stock price manipulation is costly as it is in our model. Kraft and Kühn (2009) analyse the behavior of an investor in a Black–Scholes-type market, where trading has a linear permanent impact on the stock's drift. They construct the hedging strategy and the indifference price of a European payoff for a CARA investor, and show that the optimal strategy is a combination of hedging and manipulation. In order to exploit her market impact, the investor over- or under-hedges the option, depending on his endowment and the sign of the impact term.

The second line of research our paper is connected to is stochastic differential games (see, e.g., Fleming and Soner (1993, chapter XI) and Hamadène and Lepeltier (1995) for zero sum games, Friedman (1972) and Buckdahn *et al.* (2004) for non-zero sum games or Nisio (1988) and Buckdahn and Li (2008) for viscosity solutions of Hamilton–Jacobi–Bellman (henceforth HJB) equations). The strategic interaction between large investors and its implications for market microstructure are discussed by Kyle (1985), Foster and Viswanathan (1996), Back *et al.* (2000), and Chau and Vayanos (2008), for instance. Brunnermeier and Pedersen (2005), Carlin *et al.* (2007) and Schied and Schöneborn (2007) consider predatory trading, where liquidity providers try to benefit from the liquidity demand that comes from some 'large' investor. Vanden (2005) considers a pricing game in continuous time where the option issuer controls the volatility of the underlying but does not incur liquidity costs. He derives a Nash equilibrium in the two-player, risk-neutral case and shows that "seemingly harmless derivatives, such as ordinary bull spreads, offer incentives for manipulation that are identical to those offered by digital options" (p. 1892). Closest to our setup is the paper of Gallmeyer and Seppi (2000). They consider a binomial model with three periods and finitely many risk-neutral agents holding Call options on an illiquid underlying. Assuming a linear permanent price impact and linear transaction costs, and assuming that all agents are initially endowed with the same derivative, they prove the existence of a Nash equilibrium trading strategy and indicate how market manipulation can be reduced.

A third line of research we build on is market manipulation. Different notions of market manipulations have been discussed in the literature, including short squeezes, the use of private information or false rumors (Kyle 1985, Allen and Gale 1992, Back 1992, Jarrow 1994, Pirrong 2001, Dutt and Harris 2005, Kyle and Viswanathan 2008). Most of these articles are set up in discrete time. We suggest a general mathematical framework in continuous time within which to value derivative securities in illiquid markets under strategic interactions. Specifically, we consider a stochastic differential game between a finite number of large investors ('players') holding European claims written on an illiquid stock. Their goal is to maximize the expected portfolio value at maturity, composed of trading costs and the option payoff, which depends on the trading strategies of all the other players through their impact on the dynamics of the underlying. Following Almgren and Chriss (2001) we assume that the players have a permanent impact on stock prices and that all trades are settled at the prevailing market price plus a liquidity premium. The liquidity premium can be viewed as an instantaneous price impact that affects transaction prices but not the value of the players' inventory. This form of market impact modeling is analytically more tractable than that of Obizhaeva and Wang (2005), which also allows for temporary price impacts and resilience effects. It has also been adopted by, for example, Carlin *et al.* (2007) and Schied and Schöneborn (2007) and some practitioners from the financial industry, as pointed out by Schied and Schöneborn (2008).

Our framework is flexible enough to allow for rather general liquidity costs, including the linear cost function of Almgren and Chriss (2001) and some form of bid–ask spread (see example 2.2). We show that when the market participants are risk neutral or have CARA utility functions, the pricing game has a unique Nash equilibrium. We solve the problem of equilibrium pricing using techniques from the theory of stochastic optimal control and stochastic differential games. We show that the family of the players' value functions can be characterized as the solution to a coupled system of nonlinear PDEs. Coupled systems of nonlinear PDEs arise naturally in differential stochastic games. Since general existence and uniqueness of solution results for systems of nonlinear PDEs on unbounded state spaces are hard to prove, much of the literature on stochastic differential games is confined to bounded state spaces (see, e.g., the seminal paper of Friedman (1972)). We prove an *a priori* estimate for Nash equilibria. More precisely, we prove that, under rather mild conditions, any equilibrium trading strategy is uniformly bounded. This allows us to prove that the PDE system that describes the equilibrium dynamics has a unique classical solution. The equilibrium problem can be solved in closed form for a specific market environment, namely the linear cost structure and risk-neutral agents.

It is important to know which measures may reduce market manipulation. For instance, Dutt and Harris (2005) propose position limits and Pirrong (2001) suggests efficient contract designs. We use the explicit solution for risk-neutral investors to show when 'punching the close' is not beneficial. For instance, no manipulation occurs in zero-sum games, i.e. in a game between an option writer and an option issuer. In our model, manipulation decreases with the number of informed liquidity providers and with the number of competitors, if the product is split between them. Furthermore, we find that the bid–ask spread is an important determinant of market manipulation. It turns out that the higher the spread, the less beneficial is market manipulation: high spread crossing costs make trading more costly and hence discourage frequent re-balancing of portfolio positions.

The paper is organized as follows. We present the market model as well as the optimization problem and some *a priori* estimates in section 2. The solutions for risk-neutral and CARA investors are given in sections 3 and 4, respectively. We use the explicit solution for the risk-neutral case in section 5 to show how market manipulation can be reduced.

2. The model

We adopt the market impact model of Schied and Schöneborn (2007) with a finite set of *agents* or *players* trading a single stock whose price process depends on the agents' trading strategies. Following Almgren and Chriss (2001) we shall assume that the players have a permanent impact on asset prices. All trades are settled at prevailing market prices plus a liquidity premium which depends on the *change* in the players' portfolios. In order to be able to capture changes in portfolio positions in an analytically tractable way, we assume that the stock holdings of player $j \in \{1, \ldots, N\}$ are governed by the SDE

$$\mathrm{d}X^j(s) = u^j(s)\,\mathrm{d}s, \quad X^j(0) = 0,$$

where the trading speed $u^j = \dot{X}^j$ is chosen from the following set of admissible controls, for $t \in [0, T]$:

$$\mathcal{U}_t \triangleq \{u : [t, T] \times \Omega \to \mathbb{R} \text{ progressively measurable}\}.$$

There is an array of large investor models that assume that stock holdings are absolutely continuous and that the price dynamics depend on the *change* of the investors' positions (e.g., Almgren *et al.* (2005), Almgren and Lorenz (2007), Carlin *et al.* (2007), Schied and Schöneborn (2007, 2008) and Rogers and Singh (2010)). In all the latter papers the assumption of absolute continuity is made merely for analytical convenience.

2.1. Price dynamics and the liquidity premium

Our focus is on optimal manipulation strategies (in the sense of 'punching the close') for derivatives with short maturities under strategic market interactions. For short trading periods we deem it appropriate to model the *fundamental stock price*, i.e. the value of the stock in the absence of any market impact, as a Brownian Motion with volatility $\sigma > 0$. Market impact is accounted for by assuming that the investors' accumulated stock holdings $\sum_{i=1}^N X^i$ have a linear permanent impact on the stock process P so that, for $s \in [0, T]$,

$$P(s) = P(0) + \sigma B(s) + \lambda \sum_{i=1}^{N} X^i(s), \tag{2.1}$$

with an impact parameter $\lambda > 0$. The linear permanent impact is consistent with the work of Huberman and Stanzl (2004), who argue that linearity of the permanent price impact is important to exclude quasi-arbitrage.

A trade at time $s \in [0, T]$ is settled at a *transaction price* $\tilde{P}(s)$ that includes an additional instantaneous price impact, or *liquidity premium*. Specifically,

$$\tilde{P}(s) = P(s) + g\left(\sum_{i=1}^{N} u^i(s)\right), \tag{2.2}$$

with a *cost function* g that depends on the instantaneous change $\sum_{i=1}^N u^i$ in the agents' position in a possibly nonlinear manner. The liquidity premium accounts for limited available liquidity, transaction costs, fees or spread crossing costs (see example 2.2).

Remark 1 : In our model the liquidity costs are the same for all traders and depend only on the aggregate demand throughout the entire set of agents. This captures situations where the agents trade through a market maker or clearing house that reduces the trading costs by collecting all orders and matching incoming demand and supply prior to settling the outstanding balance $\sum_{i=1}^N u^i(s)$ at market prices.

We assume that g is normalized, $g(0) = 0$ and smooth. The following additional mild assumptions on g will guarantee that the equilibrium pricing problem has a solution for risk-neutral and CARA investors.

Assumption 2.1 :

- *The derivative g' is bounded away from zero, that is $g' > \varepsilon > 0$.*
- *The mapping $z \mapsto g(z) + zg'(z)$ is strictly increasing.*

The first assumption is a technical condition needed in the proof of proposition 2.5. It appears to be not too restrictive for a cost function. Since the liquidity costs associated with a net change in the overall position z is given by $zg(z)$, the second assumption states that the agents face increasing marginal costs of trading.

Example 2.2: Among the cost functions that satisfy assumption 2.1 are the linear cost function $g(z) = \kappa z$ with $\kappa > 0$ and cost functions of the form

$$g(z) = \kappa z + c\frac{2}{\pi}\arctan(Cz), \quad \text{with } c, C > 0.$$

The former is the cost function associated with a block-shaped limit order book. The latter can be viewed as a smooth approximation of the map $z \mapsto \kappa z + c \cdot \mathrm{sign}(z)$, which is the cost function associated with a block-shaped limit order book and bid–ask spread $c > 0$.

2.2. The optimization problem

Each agent is initially endowed with a contingent claim $H^j = H^j(P(T))$, whose payoff depends on the stock price at maturity. Our focus is on optimal trading strategies in the stock, given an initial endowment. As in Gallmeyer and Seppi (2000) and Kraft and Kühn (2009), we assume that the agents do not trade the option in $[0, T]$.

To the best of our knowledge, a consistent model for trading an illiquid option with an illiquid underlying in a multi-player framework in continuous time is not available. Our work might be considered a step in this direction. We assume that the functions H^j are smooth and bounded with bounded derivatives H^j_p. This is needed in the *a priori* estimates as well as in the proof of existence of a smooth solution to the HJB equation.

Remark 2: We only consider options with cash settlement. This assumption is key. While cash settlement is susceptible to market manipulation, we show in proposition 5.5 below that when deals are settled physically, i.e. when the option issuer delivers the underlying, market manipulation is not beneficial: Any price increase is outweighed by the liquidity costs of subsequent liquidation. We note that this only applies to 'punching the close'. There are other types of market manipulation, such as corners and short squeezes, which might be beneficial when deals are settled physically, but which are not captured by our model (see Jarrow (1994) or Kyle and Viswanathan (2008)).

We shall now give a heuristic derivation of the optimization problem. Consider a single risk-neutral investor who builds up a position in stock holdings $X(T)$ using the trading strategy u in $[0, T]$ and afterwards liquidates his stock holdings using a constant rate of liquidation η, so that, at time $T' \triangleq T + |X(T)/\eta|$, the portfolio is liquidated. In view of (2.2), the proceeds from such a round trip strategy are

$$\begin{aligned}\int_0^T -u(s)\tilde{P}(s)\,\mathrm{d}s + \int_T^{T'} \eta\tilde{P}(s)\,\mathrm{d}s \\ = \int_0^{T'} \sigma B(s)\,\mathrm{d}X(s) - \lambda\int_0^{T'} X(s)\,\mathrm{d}X(s) \\ - \int_0^T u(s)\,g(u(s))\,\mathrm{d}s - X(T)\,g(\eta).\end{aligned}$$

Using integration by parts and $X(0) = X(T') = 0$ we see that the first term on the second line has zero expectation† and the second term also vanishes. The last term describes the liquidity costs of the constant liquidation rate η and goes to zero if η goes to zero since $g(0) = 0$. In this sense, infinitely slow liquidation incurs no costs. It follows that the round trip strategy described above incurs expected liquidity costs of $-\int_0^T u(s)\,g(u(s))\,\mathrm{d}s$. Taking into account the option payoff, the optimization problem for a single risk-neutral investor becomes

$$\sup_{u\in\mathcal{U}_0} \mathbb{E}\left[-\int_0^T u(s)\,g(u(s))\,\mathrm{d}s + H(P(T))\right]. \tag{2.3}$$

This reflects the trade-off between liquidity costs (the costs of 'punching the close') and an increased‡ option payoff. Unfortunately, this heuristic derivation has no direct counterpart in the multi-player case. As one prerequisite, one would need the optimal liquidation strategies (and corresponding liquidation value) of several agents in a market with general liquidity structure. Defining a notion of liquidation value under strategic interaction is still an open question (Carlin *et al.* (2007) and Schied and Schöneborn (2007) derived solutions in special cases) and it is not the focus of this paper. Our focus is on the trade-off between increased option payoff and liquidity costs in a multi-player framework. Specifically, we assume that the preferences of player j at time $t \in [0, T]$ are described by a preference functional Ψ^j_t (conditional expected value or conditional entropic risk measure) and that his goal at time $t = 0$ is to maximize the utility from the option payoff minus the cost of trading (given the other players' strategies). We hence consider the following optimization problem.

Problem 2.3: Given the strategies $u^i \in \mathcal{U}_0$ for all the players $i \neq j$, the optimization problem of player $j \leq N$ is

$$\sup_{u^j\in\mathcal{U}_0} \Psi^j_0\left(-\int_0^T u^j(s)\,g\left(\sum_{i=1}^N u^i(s)\right)\mathrm{d}s + H^j(P(T))\right).$$

The case where all investors are risk neutral, $\Psi^j_t(Z) = \mathbb{E}[Z \mid \mathcal{F}_T]$, is studied in section 3. The case of conditional expected exponential utility maximizing investors is studied in section 4. In that case we may choose $\Psi^j_t(Z) = -(1/\alpha^j)\log\mathbb{E}[\exp(-\alpha^j Z) \mid \mathcal{F}_T]$, where $\alpha^j > 0$ denotes the risk aversion of player j. Both preference functionals are *translation invariant*.§ This means that $\Psi^j_t(Z + Y) = \Psi^j_t(Z) + Y$ for any random variable Y that is measurable with respect to the information available at time $t \in [0, T]$. As a result, the trading costs incurred up to time t do not affect the optimal trading strategy at later times. This property is key and will allow us to establish the existence of Nash equilibria in our financial market model.

Definition 2.4: We say that a vector of strategies $(u^1, \ldots, u^N)$ is a *Nash equilibrium* if, for each agent $j \leq N$, his trading strategy u^j is a best response against the behavior of all the other players, i.e. if u^j solves problem 2.3, given the other players' aggregate trading $u^{-j} \triangleq \sum_{i\neq j} u^i$.

Remark 3: Our results hinge on two key assumptions: the restriction to absolutely continuous trading strategies and the focus on the trade-off between trading costs and market manipulation. Both restrictions may be considered undesirable. On the other hand, strategies with absolutely continuous and jump parts in continuous time

†We will prove an *a priori* estimate in proposition 2.5 and then only consider bounded strategies, so that the stochastic integral $\int_0^T X(t)\,\mathrm{d}B(t)$ is indeed a martingale.

‡The only purpose of trading is an increased option payoff and not, for instance, hedging. For a study of the interplay of hedging and manipulation we refer the reader to Kraft and Kühn (2009).

§Translation invariant preferences have recently attracted much attention in the mathematical finance literature in the context of optimal risk sharing and equilibrium pricing in dynamically incomplete markets. We refer to Cheridito *et al.* (2009) for further details.

would call for methods of singular or impulse control, viscosity solutions, and (systems of) quasi-variational inequalities. This, as well as introducing a notion of a liquidation value under strategic interaction, is well beyond the scope of this article. Our model should instead be viewed as a first benchmark to more sophisticated models. Despite its many simplifications it yields some insight into the qualitative behavior of optimal manipulation strategies as well as 'rules of thumb' for traders or regulators. Moreover, our approach allows for explicit solutions, which will be used in section 5 to indicate how manipulation can be reduced.

2.3. A priori estimate

In the sequel we show that problem 2.3 admits a unique solution for risk-neutral and CARA investors. The proof uses the following *a priori* estimates for the optimal trading strategies. It states that, if an equilibrium exists, then each player's trading speed is bounded. The reason is that the derivatives H_p^j of the payoff functions H^j are assumed to be bounded, so each investor benefits at most linearly from fast trading. However, trading costs grow more than linearly, and thus very fast trading is not beneficial. Note that this result does not depend on the preference functional.

Proposition 2.5: *Let* $(u^1, \ldots, u^N)$ *be a Nash equilibrium for problem* 2.3. *Then each strategy* u^j *satisfies* $dt \times d\mathbb{P}$ *a.e.*

$$|u^j(s)| \le N\frac{\lambda}{\varepsilon}\left(\max_{i\le N}\|H_p^i\|_\infty + 1\right),$$

where ε *is taken from assumption* 2.1.

Proof: Let $j \le N$, $h \triangleq \max_i \|H_p^i\|_\infty$ and

$$A \triangleq \left\{(s,\omega) \in [0,T]\times\Omega : \sum_{i=1}^N u^i(s,\omega) \ge 0\right\}$$

be the set where the aggregate trading speed is non-negative. Let us fix the sum of the competitors' strategies u^{-j}. On the set A the best response $u^j(s)$ is bounded from above by $K \triangleq (\lambda/\varepsilon)(h+1)$. Otherwise, the truncated strategy $\bar{u}^j(s) \triangleq u^j(s) \wedge K\mathbb{1}_A + u^j(s)\mathbb{1}_{A^c}$ would outperform $u^j(s)$. To see this, let us compare the payoffs associated with u^j and $\bar{u}^j$. We denote by $P^{\bar{u}^j}(T)$ and $P^{u^j}(T)$ the stock price under the strategies $\bar{u}^j$ and u^j, respectively. The payoff associated with $\bar{u}^j$ minus the payoff associated with u^j can be estimated from below as

$$\begin{aligned}
&-\int_0^T \bar{u}^j(s)\, g(\bar{u}^j(s) + u^{-j}(s))\,\mathrm{d}s + H^j(P^{\bar{u}^j}(T)) \\
&\quad + \int_0^T u^j(s)\, g(u^j(s) + u^{-j}(s))\,\mathrm{d}s - H^j(P^{u^j}(T)) \\
&\quad \ge \int_0^T \bar{u}^j(s)\big(g(u^j(s)+u^{-j}(s)) - g(\bar{u}^j(s)+u^{-j}(s))\big)\,\mathrm{d}s \\
&\quad + \int_0^T (u^j(s) - \bar{u}^j(s))\, g(u^j(s)+u^{-j}(s))\,\mathrm{d}s - \lambda(X^j(T) \\
&\quad - Y^j(T))\|H_p\|_\infty.
\end{aligned}$$

Note that $u^j(s) + u^{-j}(s) \ge 0$ on A and thus $g(u^j(s)+u^{-j}(s)) \ge 0$ due to assumption 2.1. Furthermore, $g(u^j(s)+u^{-j}(s)) - g(\bar{u}^j(s)+u^{-j}(s)) \ge \varepsilon(u^j(s) - \bar{u}^j(s))$, again by assumption 2.1. The difference in the payoffs is therefore larger than

$$\begin{aligned}
&\int_0^T \bar{u}^j(s)\varepsilon(u^j(s) - \bar{u}^j(s))\,\mathrm{d}s - \lambda h\int_0^T (u^j(s) - \bar{u}^j(s))\,\mathrm{d}s \\
&\quad = \int_{u^j(s)>\bar{u}^j(s)} (\varepsilon\bar{u}^j(s) - \lambda h)(u^j(s) - \bar{u}^j(s))\,\mathrm{d}s.
\end{aligned}$$

On the set $\{u^j(s) > \bar{u}^j(s)\}$ we have $\bar{u}^j(s) = K = (\lambda/\varepsilon)(h+1)$ and the above expression is strictly positive, a contradiction. This shows that $u^j(s)$ is bounded above by K on the set A for each $j \le N$. Still on the set A, we obtain the following lower bound:

$$u^j(s) = \sum_{i=1}^N u^i(s) + \sum_{i\ne j} -u^i(s) \ge 0 - (N-1)K. \qquad (2.4)$$

A symmetric argument on the set $B \triangleq \{(s,\omega) \in [0,T]\times\Omega : \sum_{i=1}^N u^i(s,\omega) \le 0\}$ completes the proof. □

3. Solution for risk-neutral investors

In this section we use dynamic programming to show that problem 2.3 admits a unique solution (in a certain class) for risk-neutral agents. Here the preference functional is $\Psi_t^j(Z) = \mathbb{E}[Z \mid \mathcal{F}_T]$ for each $j \le N$. We also show that the solution can be given in closed form for the special case of a linear cost function.

The idea is to consider the value function associated with problem 2.3 for player j, where his competitors' strategies are fixed, and to characterize it as the solution of the HJB PDE. Solving the resulting coupled system of PDEs for all players simultaneously then provides an equilibrium point of the stochastic differential game (Friedman 1972). To begin with, we fix the strategies $(u^i)_{i\ne j}$ and define the *value function* for player $j \le N$ as

$$V^j(t,p) = \sup_{u^j\in\mathcal{U}_t} \mathbb{E}_{t,p}\left[-\int_t^T u^j(s)\, g\left(\sum_{i=1}^N u^i(s)\right)\mathrm{d}s + H^j(P(T))\right],$$

subject to the state dynamics

$$\mathrm{d}P(s) = \sigma\,\mathrm{d}B(s) + \lambda\sum_{i=1}^N u^i(s)\,\mathrm{d}s, \qquad P(t) = p.$$

Here we use the notation $\mathbb{E}_{t,p}[\cdot] \triangleq \mathbb{E}[\cdot \mid P_t = p]$. Given time $t \in [0,T]$ and stock price $p \in \mathbb{R}$ the value function represents the conditional expected portfolio value at maturity that player j can achieve by trading optimally, given the other players' strategies. The associated HJB equation is (Fleming and Soner 1993, theorem IV.3.1)

$$0 = v_t^j + \frac{1}{2}\sigma^2 v_{pp}^j + \sup_{c^j\in\mathbb{R}}[\lambda(c^j + u^{-j})v_p^j - c^j g(c^j + u^{-j})], \qquad (3.1)$$

with terminal condition $v^j(T, p) = H^j(p)$, where v_t and v_p denote time and spatial derivatives, respectively. The HJB equation is formulated in terms of the *candidate* value functions $v^1, \ldots, v^N$ instead of the actual value functions $V^1, \ldots, V^N$. We first need to show existence and uniqueness of a smooth solution to (3.1) before we can identify v^i with V^i. Given the aggregate trading strategy u^{-j} of all the other agents, a candidate for the maximizer $c^j = u^j$ in (3.1) should satisfy

$$0 = \lambda v_p^j - g(c^j + u^{-j}) - c^j g'(c^j + u^{-j}). \qquad (3.2)$$

We have one equation of this type for each player $j \le N$. Summing them up and defining the *aggregate trading speed* as

$$u^* \triangleq \sum_{i=1}^{N} u^i$$

yields the following condition:

$$0 = \lambda \sum_{i=1}^{N} v_p^i - Ng\left(\sum_{i=1}^{N} u^i(s)\right) - \left(\sum_{i=1}^{N} u^i(s)\right) g'\left(\sum_{i=1}^{N} u^i(s)\right)$$
$$= \lambda \sum_{i=1}^{N} v_p^i - Ng(u^*(s)) - u^*(s) g'(u^*(s)). \qquad (3.3)$$

In view of assumption 2.1 the map $z \mapsto N\, g(z) + zg'(z)$ is strictly increasing. Hence condition (3.3) admits a unique solution u^* that depends on $\sum_{i=1}^{N} v_p^i$. Plugging u^* back into (3.2) allows us to compute the candidate optimal control for player $j \le N$ as

$$c^j = u^j = \frac{\lambda v_p^j - g(u^*)}{g'(u^*)}. \qquad (3.4)$$

This expression is well defined since $g' > 0$, again by assumption 2.1. Plugging this candidate optimal control into the HJB equation, we see that the system of HJB PDEs now takes the form

$$0 = v_t^j + \frac{1}{2}\sigma^2 v_{pp}^j + \lambda\left(u^* - \frac{g(u^*)}{g'(u^*)}\right) v_p^j + \frac{g(u^*)^2}{g'(u^*)}, \qquad (3.5)$$

with terminal condition $v^j(T, p) = H^j(p)$ for $j \le N$. Note that the coupling stems from the aggregate trading speed u^* via condition (3.3).

Remark 4: Looking back, we have turned the individual HJB equations (3.1) into the system of coupled PDEs (3.5). Systems of this form appear naturally in the theory of differential games, but we did not find a reference that covers this particular case. Theorem 1 of Friedman (1972), for instance, is valid only on a bounded state space. We shall use our *a priori* estimate of proposition 2.5 in order to prove the existence of a unique solution to (3.5).

The following theorem, the proof of which is given in appendix A, shows that the system of PDEs (3.5) has a unique classical solution if $H^j \in \mathcal{C}_b^2$, i.e. H^j is twice continuously differentiable and its derivatives up to order 2 are bounded, for each j. Similarly, $\mathcal{C}^{1,2}$ is the space of functions that are continuously differentiable in time and twice continuously differentiable in space.

Theorem 3.1: *Let $H \in \mathcal{C}_b^2$. Then the Cauchy problem* (3.5) *admits a unique classical solution in $\mathcal{C}^{1,2}$, which is the vector of value functions.*

Remark 5 : An alternative way of solving the system (3.5) is the following: If we sum up the N equations, we obtain a Cauchy problem for the aggregate value function $v \triangleq \sum_{i=1}^{N} v^i$, namely

$$0 = v_t + \frac{1}{2}\sigma^2 v_{pp} + u^*[\lambda v_p - g(u^*)], \qquad (3.6)$$

with terminal condition $v(T,p) = \sum_{i=1}^{N} H^i(p)$. The existence and uniqueness of a solution to this one-dimensional problem can be shown using theorem V.8.1 of Ladyzenskaja *et al.* (1968). Once the solution is known, we can plug it back into (3.5) and obtain N decoupled equations. This technique is applied in the following section where we construct an explicit solution for linear cost functions.

It is hard to find a closed-form solution for the coupled PDE (3.5). However, for the particular choice $g(z) = \kappa z$ with a liquidity parameter $\kappa > 0$ the solution to (3.5) can be given explicitly. Here and throughout, we denote by

$$f_{\mu,\sigma^2}(z) \triangleq \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(z-\mu)^2}{2\sigma^2}\right)$$

the normal density with mean μ and variance σ^2.

Proposition 3.2: *Let $g(z) = \kappa z$. Then the solution of* (3.5) *can be given in closed form as the solution to a non-homogeneous heat equation.*

Proof: The optimal trading speed from (3.4) and the aggregate trading speed from (3.3) are

$$u^j = \frac{\lambda}{\kappa}\left(v_p^j - \frac{1}{N+1}\sum_{i=1}^{N} v_p^i\right), \qquad (3.7)$$

$$u^* = \sum_{i=1}^{N} u^i = \frac{\lambda}{\kappa(N+1)} \sum_{i=1}^{N} v_p^i = \frac{\lambda}{\kappa(N+1)} v_p. \qquad (3.8)$$

Equation (3.5) for player j's value function now becomes

$$0 = v_t^j + \frac{1}{2}\sigma^2 v_{pp}^j + \kappa(u^*)^2.$$

Combining this with (3.8) and summing up for $j = 1, \ldots, N$ yields the following PDE for the aggregate value function $v = \sum_{i=1}^{N} v^i$:

$$0 = v_t + \frac{1}{2}\sigma^2 v_{pp} + \frac{\lambda^2 N}{\kappa(N+1)^2} v_p^2, \qquad (3.9)$$

with terminal condition $v(T,p) = \sum_{i=1}^{N} H^i(p)$. This PDE is a variant of *Burgers' equation* (Rosencrans 1972). It allows for an explicit solution, which we cite in lemma C.1 in appendix C. With this solution at hand, we can solve for each single investor's value function. We plug the solution v back into equations (3.7) and (3.8) for

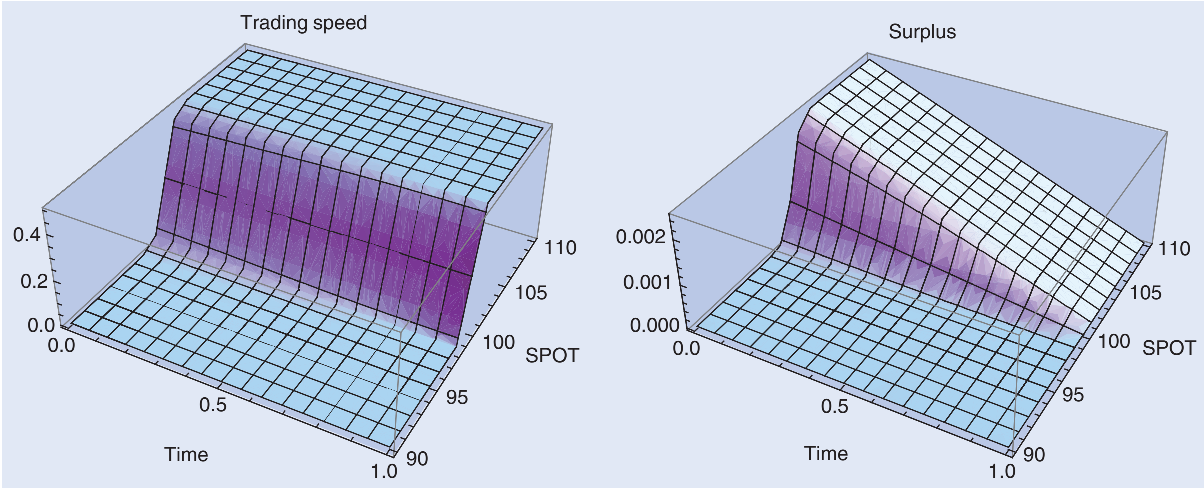

Figure 1. Trading speed and surplus for one risk-neutral investor holding a European Call option.

the trading speeds, and those into the PDE (3.5). This yields

$$0 = v_t^j + \frac{1}{2}\sigma^2 v_{pp}^j + \frac{\lambda^2}{\kappa(N+1)^2} v_p^2,$$

with terminal condition $v^j(T, p) = H^j(p)$. This is now a PDE in the unknown function v^j with known function v_p. We see that it is a non-homogeneous heat equation with solution given by

$$\begin{aligned} v^j(T-t,p) &= \int_{\mathbb{R}} H^j(z) f_{p,\sigma^2 t}(z)\,\mathrm{d}z \\ &\quad + \frac{\lambda^2}{\kappa(N+1)^2}\int_0^t \int_{\mathbb{R}} v_p^2(s,z) f_{p,\sigma^2(t-s)}(z)\,\mathrm{d}z\,\mathrm{d}s, \end{aligned}$$

where v is known from lemma C.1 (in particular, it is bounded and integrable). □

Let us conclude this section with some numerical illustrations. For risk-neutral players and a linear cost structure, we reduced the system of PDEs to the one-dimensional PDE (3.9) for the aggregate value function. This can be interpreted as the value function of the representative agent. Such reduction to a representative agent is not always possible for more general utility functions. In the sequel we illustrate the optimal trading speed $u(s, p)$ and surplus of a representative agent as functions of time and spot prices for a European call option $H(P(T)) = (P(T) - K)^+$ and digital option $H(P(T)) = \mathbb{1}_{\{P(T)\geq K\}}$, respectively.† By surplus, we mean the difference between the representative agent's optimal expected portfolio value $v(t,p)$ and the conditional expected payoff $\mathbb{E}_{t,p}[H(P(T))]$ in the absence of any market impact. It represents the expected net benefit due to price manipulation.

We choose a linear cost function, strike $K = 100$, maturity $T = 1$, volatility $\sigma = 1$ and liquidity parameters $\lambda = \kappa = 0.01$. We see from figure 1 that, for the case of a Call option, both the optimal trading speed and the surplus increase with the spot; the latter also increases with the time to maturity. Furthermore, the increase in the trading speed is maximal when the option is at the money. For digital options (figure 2) the trading speed is highest for at-the-money options close to maturity as the trader tries to push the spot above the strike. If the spot is far away from the strike, the trading speed is very small as it is unlikely that the trader can push the spot above the strike before expiry. For both option types, a high spread renders manipulation unattractive. Figures 3 and 4 show the optimal trading speed and the surplus at time $t = 0$ for the Call and Digital option for a representative agent. We use the cost function

$$\begin{aligned} g(z) = \kappa z + c \cdot \mathrm{sign}(z), &\quad \text{for different spreads} \\ c \in \{0, 0.001, 0.002, 0.003, 0.004\}, & \qquad (3.10) \end{aligned}$$

with the remaining parameters as above. We see that the higher the spread, the smaller the trading speed and the surplus. This is intuitive, as frequent trading, in particular when the option is at the money, incurs high spread crossing costs. The same is true for fixed transaction costs, which also discourage frequent trading.

4. Solution for CARA investors

In the preceding section we considered risk-neutral investors. We shall now extend the analysis of problem 2.3 to the class of entropic preference functionals with risk-aversion coefficient $\alpha^j > 0$, given by

$$\Psi_t^j(Z) = -\frac{1}{\alpha^j}\log \mathbb{E}[\exp(-\alpha^j Z) \mid \mathcal{F}_T].$$

†Note that the cost function in (3.10) is not smooth, and the Call and Digital options are not smooth and bounded, so theorem 3.1 does not apply directly. There are two ways to overcome this difficulty: We could either approximate g and H by smooth and bounded functions, or we could interpret v not as a classical, but only as a viscosity solution of (3.1) (Fleming and Soner 1993, chapter V).

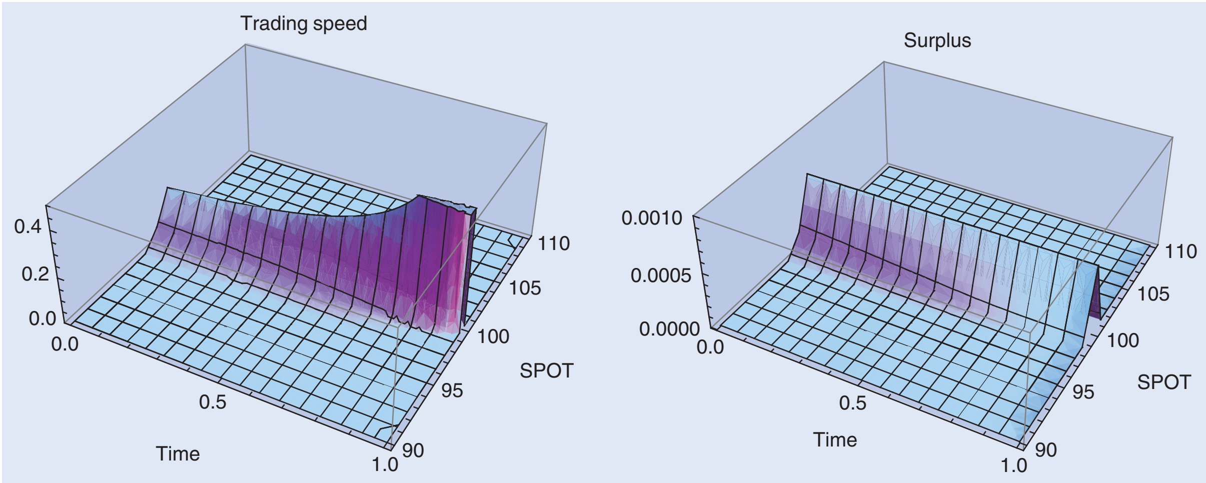

Figure 2. Trading speed and surplus for one risk-neutral investor holding a Digital option.

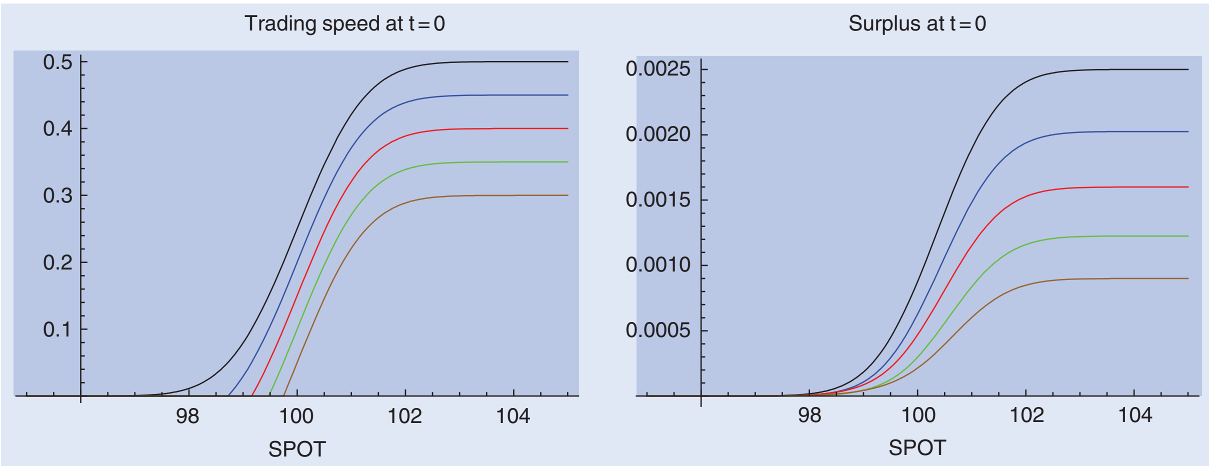

Figure 3. Trading speed and surplus for a risk-neutral investor holding a European Call option for different spread sizes $s = 0$ (black), 0.001 (blue), 0.002 (red), 0.003 (green) and 0.004 (brown). The higher the spread, the smaller the trading speed and the surplus.

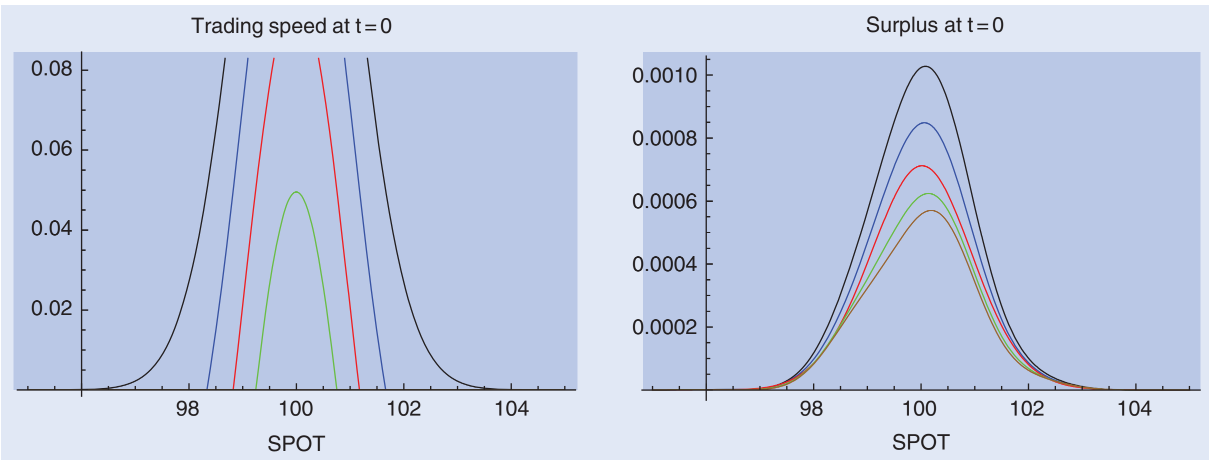

Figure 4. Trading speed and surplus for a risk-neutral investor holding a Digital option for different spread sizes $s = 0$ (black), 0.001 (blue), 0.002 (red), 0.003 (green) and 0.004 (brown). The higher the spread, the smaller the trading speed and the surplus.

As pointed out by Cheridito *et al.* (2009, p. 9), these mappings induce the same preferences as conditional expected exponential utility functions. Due to the translation invariance of Ψ_t^j, the trading costs $R(t) \triangleq \int_0^t u^j(s)\, g(\sum_{i=1}^N u^i(s))\, \mathrm{d}s$ that player j incurred in $[0, t]$ do not affect the player's optimal strategy in the time interval $[t, T]$ (they affect only the utility). As a result, we may consider the cost-adjusted preference functional

$\Psi_t^j + R(t)$. So, given the strategies $(u^i)_{i\neq j}$ of the other players, the value function for player $j \leq N$ is

$$V^j(t,p) \triangleq \sup_{u^j \in \mathcal{U}_t} \left\{ -\frac{1}{\alpha^j} \log \mathbb{E}_{t,p}\left[\exp\left(-\alpha^j \left(-\int_t^T u^j(s)\, g \left(\sum_{i=1}^N u^i(s) \right) \mathrm{d}s + H^j(P(T)) \right) \right) \right] \right\}.$$

As a result, the HJB equation† for player j is now given by

$$0 = \tilde{v}_t^j + \frac{1}{2}\sigma^2 \tilde{v}_{pp}^j - \frac{1}{2}\sigma^2 \alpha^j (\tilde{v}_p^j)^2 + \sup_{c^j \in \mathbb{R}} [\lambda(c^j + u^{-j})\tilde{v}_p^j - c^j g(c^j + u^{-j})], \tag{4.1}$$

with terminal condition $\tilde{v}^j(T,p) = H^j(p)$. Note that this equation equals the HJB equation (3.1) in the risk-neutral setting, up to the quadratic term $-\frac{1}{2}\sigma^2\alpha^j(\tilde{v}_p^j)^2$. Applying the same arguments as in section 3, the candidate optimal trading speeds are, for $j \leq N$,

$$c^j = u^j = -\frac{1}{g'(u^*)}[-\lambda \tilde{v}_p^j + g(u^*)],$$

where the aggregate trading speed u^* is the unique solution to

$$0 = \lambda \sum_{i=1}^N \tilde{v}_p^i - Ng\left(\sum_{i=1}^N u^i(s) \right) - \left(\sum_{i=1}^N u^i(s) \right) g'\left(\sum_{i=1}^N u^i(s) \right). \tag{4.2}$$

If we plug u^* and u^j back into (4.1), we obtain

$$0 = \tilde{v}_t^j + \frac{1}{2}\sigma^2 \tilde{v}_{pp}^j - \frac{1}{2}\sigma^2\alpha^j(\tilde{v}_p^j)^2 + \lambda\left(u^* - \frac{g(u^*)}{g'(u^*)} \right)\tilde{v}_p^j + \frac{g(u^*)^2}{g'(u^*)}. \tag{4.3}$$

We can show existence and uniqueness of a solution.

Theorem 4.1 : *Let $H^j \in \mathcal{C}_b^2$ for each $j \leq N$. The Cauchy problem* (4.1) *admits a unique solution, which is the vector of value functions.*

Proof: See appendix A. □

For the one-player case with linear cost structure, we have an explicit solution.

Corollary 4.2: *Let $N = 1$ and $g(z) = \kappa z$. Then the Cauchy problem* (4.1) *admits a unique solution that can be given in closed form.*

Proof: The maximizer in (4.1) is now

$$c = u = \frac{\lambda}{2\kappa}\tilde{v}_p,$$

and the Cauchy problem (4.3) turns into

$$0 = \tilde{v}_t + \frac{1}{2}\sigma^2\tilde{v}_{pp} + \left(\frac{\lambda^2}{4\kappa} - \frac{1}{2}\sigma^2\alpha \right)\tilde{v}_p^2,$$

with terminal condition $\tilde{v}(T,p) = H(p)$. This is Burgers' equation. Its explicit solution is given in lemma C.1 in appendix C. □

Let us conclude this section with numerical illustrations for the two-player case. Figure 5 shows the aggregate optimal trading speed and the surpluses $v^j(0,p) - \Psi_0^j(H^j(P(T)))$ for time $t = 0$ and different spot prices $p \in [90, 100]$ for the European Call option $H(P(T)) = (P(T) - K)^+$. We assume that player 1 (blue) is the option writer and player 2 (red) the option issuer. We chose the strike $K = 100$, maturity $T = 1$, volatility $\sigma = 1$ and liquidity parameters $\lambda = 0.1$, $\kappa = 0.01$ and risk-aversion parameters $\alpha^1 = 0.01$, $\alpha^2 = 0.01$ (solid) and $\alpha^1 = 0.1$, $\alpha^2 = 0.001$ (dashed). Since player 1 has a long position in the option, he has an incentive to buy the underlying; for the same reason, player 2 has an incentive to sell it (panel (b)). Our simulations suggest that the option issuer is slightly more active than the option writer, in particular near the strike. Furthermore, we see from panel (d) that the issuer benefits more from reducing his loss than the writer benefits from increasing his gains; this effect is due to the concavity of the utility function. If the option issuer is less risk averse than the option writer, he trades and benefits slightly more (dashed).

Figure 6 shows the same plots for the Digital option. Now the option writer trades faster and benefits more if the option is in the money, while the issuer trades faster and gains more if the option is out of the money (panels (c) and (d)).

5. How to reduce manipulation

In this section, we use the results for risk-neutral agents derived in section 3 to illustrate how an option issuer may prevent‡ other market participants from trading against him by using their impact on the dynamics of the underlying. Some of our observations have already been reported by Kumar and Seppi (1992) for Futures in a two-period model and by Gallmeyer and Seppi (2000) for Call options in a three-period binomial model. Note that the results of this section only hold for risk-neutral investors.

As a first step, we show that market manipulation is not beneficial if traders have no permanent impact on the price of the underlying.

Proposition 5.1: *If $\lambda = 0$, then $u^j \equiv 0$ for each $j \leq N$.*

†This PDE can be derived by considering the exponential utility function first and then applying a logarithmic transformation. In this approach, it is necessary to introduce new state variables that keep track of each agent's trading costs. Due to translation invariance, these variables factor out and can be dropped again.

‡Let us emphasize again that our results only apply to the practice of 'punching the close', i.e. manipulating the stock price in order to increase a given option payoff. There are other types of market manipulation not covered by our setup, such as market corners, short squeezes, the use of private information or false rumors. We refer the interested reader to Jarrow (1994) and Kyle and Viswanathan (2008).

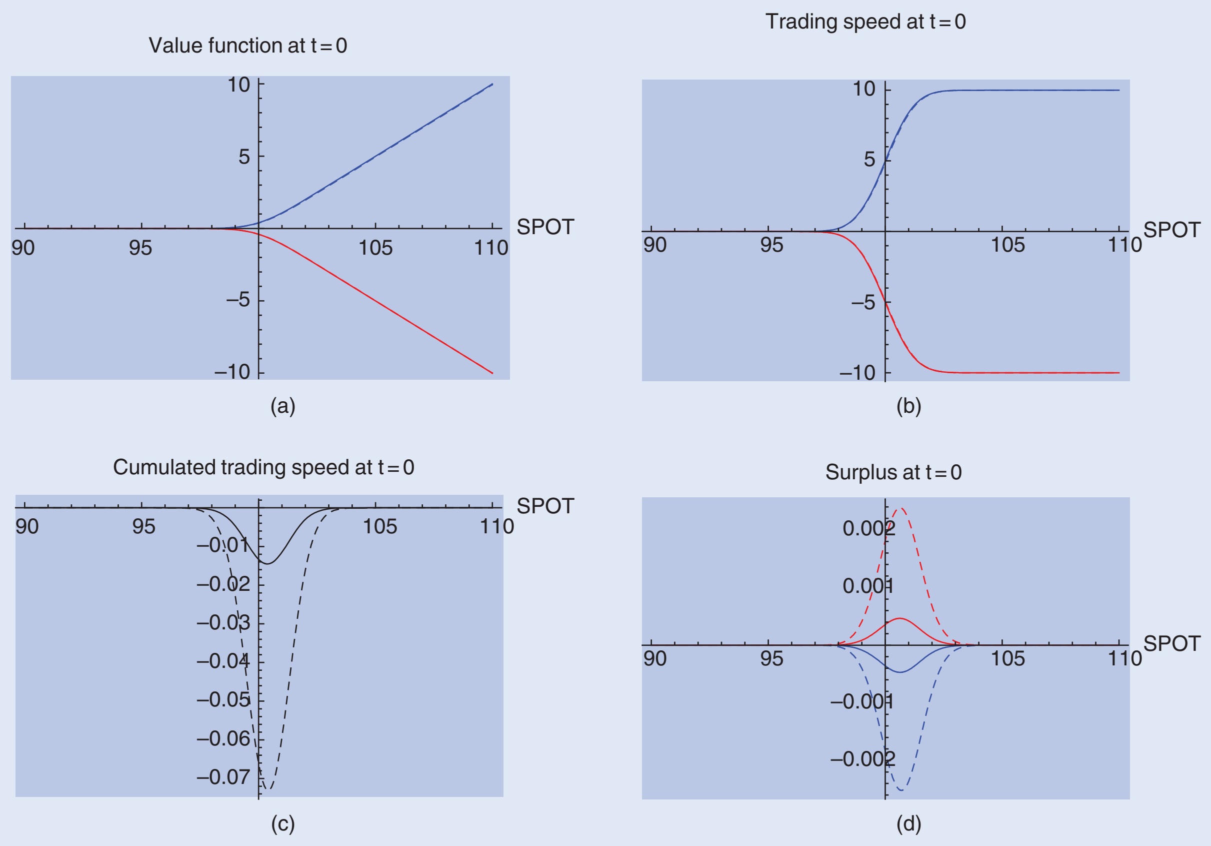

Figure 5. Value function, trading speed, aggregate trading speed and surplus for the writer (blue) and issuer (red) of a European Call option when both agents are risk averse. The solid (dashed) curves display the case where the issuer is about as (less) risk averse as (than) the option writer.

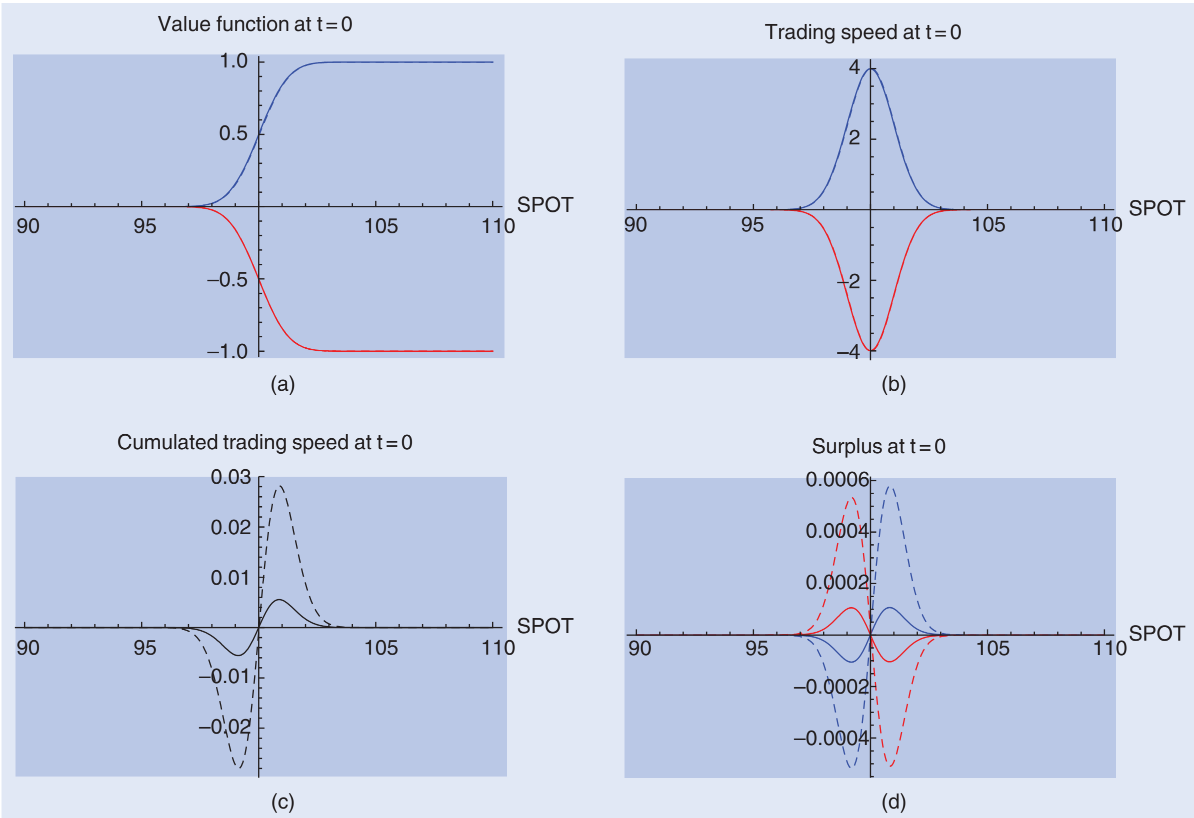

Figure 6. Value function, trading speed, aggregate trading speed and surplus for the writer (blue) and issuer (red) of a European Digital option when both agents are risk averse. The solid (dashed) curves display the case where the issuer is about as (less) risk averse as (than) the option writer.

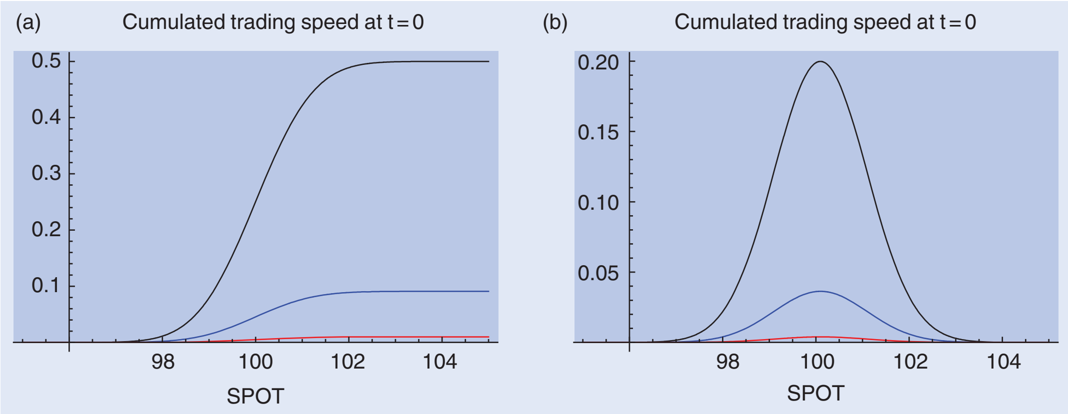

Figure 7. Aggregate trading speed u^* at time $t=0$ for $N=1$ (black), 10 (blue) and 100 (red) players each holding $1/N$ shares of a Call (left) and Digital (right) option with strike $K=100$. The more agents, the less aggregate manipulation.

Proof: First note that $u^* = \sum_{i=1}^{N} u^i = 0$ is the unique solution to (3.3). Now (3.4) implies that $u^j \equiv 0$ for each $j \leq N$. □

Let us now consider the more interesting case of $\lambda > 0$. We show next that, in the case of offsetting payoffs, the aggregate trading speed is zero. Put differently, in a zero-sum game of risk-neutral investors willing to move the market in their favor, their combined effect cancels. We note that this is no longer true for general utility functions, as illustrated in figure 5 for the CARA case.

Proposition 5.2: *If* $\sum_{i=1}^{N} H^i = 0$, *then* $\sum_{i=1}^{N} u^i \equiv 0$.

Proof: Consider the PDE (3.6) for the aggregate value function with terminal condition zero and the characterization (3.3) of the aggregate trading speed. $u^* = \sum_{i=1}^{N} u^i \equiv 0$ and $v = \sum_{i=1}^{N} v^i \equiv 0$ is the unique solution to this coupled system. □

In reality, some (or all) of the investors might not want to manipulate, e.g. for legal reasons.† This is why we now look at the following asymmetric situation: The option issuer, player 0, does not trade the underlying; his competitor, player 1, owns the payoff $H^1 \neq 0$ and intends to move the stock price in his favor. In addition, there are $N-1$ informed investors without option endowment in the market. They are 'predators' that may supply liquidity and thus reduce the first player's market impact (Carlin *et al.* 2007, Schied and Schöneborn 2007). The following result states that the aggregate trading speed is decreasing in the number of players. More liquidity suppliers lead to more competition for profit and less (cumulated) market manipulation. If the number of players goes to infinity, manipulation vanishes. Note that propositions 5.3 and 5.4 are only valid for the linear cost function, as the proofs hinge on the closed-form solution obtained in proposition 3.2, and for non-decreasing payoff functions.

Proposition 5.3: *Let* $g(z)=\kappa z$. *Let* $H^1 \in \mathcal{C}_b^2$ *be non-decreasing and* $H^i = 0$ *for* $i=2,\ldots,N$. *Then for* $s \in [0,T]$ *the aggregate trading speed* $\sum_{i=1}^{N} u^i(s)$ *is decreasing in N and*

$$\lim_{N\to\infty} \sum_{i=1}^{N} u^i(s) = 0.$$

Proof: See appendix B. □

Let us modify the preceding setting a little. Again, player 0 issues a product H and does not intend to manipulate the underlying, while his competitors do. More precisely, assume that player 0 splits the product H into pieces and sells them to N risk-neutral competitors, such that each of them gets $(1/N)H$. We find that their aggregate trading speed $\sum_{i=1}^{N} u^i$ is decreasing in the number of competitors N. Consequently, the option issuer should sell his product to as many investors as possible in order not to be susceptible to manipulation. We illustrate this result in figure 7, which shows the aggregate trading speed at time $t=0$ of N players each holding $1/N$ option shares.

Proposition 5.4: *Let* $g(z)=\kappa z$. *Let* $H \in \mathcal{C}_b^2$ *be non-decreasing and* $H^i = (1/N)H$ *for* $i=1,\ldots,N$. *Then for* $s \in [0,T]$ *the aggregate trading speed* $\sum_{i=1}^{N} u^i(s)$ *is decreasing in N and*

$$\lim_{N\to\infty} \sum_{i=1}^{N} u^i(s) = 0.$$

Proof: See appendix B. □

The preceding results indicate how an option issuer can prevent his competitors from manipulation. One strategy is public announcement of the transaction: the more informed liquidity suppliers that are on the market, the smaller the impact on the underlying. A second strategy is splitting the product into pieces—the more option writers, the less manipulation. Let us conclude this section with a surprisingly simple way to avoid manipulation: using options with physical settlement. In contrast to cash

†A discussion of legal issues is beyond the scope of this paper, but see the discussion of Kyle and Viswanathan (2008).

settlement, the option holder does not receive (pay) the current price of the underlying, but receives (delivers) stock shares. In the case of Call options, for instance, let us denote by c^j the number of Calls player j decides to execute at maturity; he then holds $X^j(T)+c^j$ stock shares whose liquidation value under infinitely slow liquidation in $[T,\infty)$ is now defined as

$$(X^j(T)+c^j)\left(P(T)-\frac{1}{2}\lambda(X^j(T)+c^j)\right).$$

The following proposition shows that, in a framework of several risk-neutral players holding physically settled Calls, Puts and Forwards, it is optimal not to manipulate the underlying.

Proposition 5.5: *Consider N risk-neutral agents holding European Call, Put or Forward options with physical settlement. Then $u^j\equiv 0$ for each $j\leq N$ is a Nash equilibrium.*

Proof: We only prove the assertion for Call options. The case of Puts and Forwards (or combinations thereof) follows by the same arguments. Suppose that agent $j\leq N$ is endowed with $C^j\geq 0$ Call options with physical settlement and strike K^j. At maturity, the agent decides how many options he exercises. The agent's strategy is now a pair (u^j, c^j), where $u^j\in\mathcal{U}_0$ denotes his trading speed in the underlying and $c^j\in[0, C^j]$ the number of Call options exercised. At maturity, the agent receives c^j stock shares for the price $c^j K^j$. Suppose that $u^i\equiv 0$ for each $i\neq j$, i.e. none of player j's competitors trades. His optimization problem is then

$$\sup_{u^j,c^j}\mathbb{E}\left[\int_0^T -u^j(s)\tilde{P}(s)\,\mathrm{d}s - c^jK^j + (X^j(T)+c^j)\times\left(P(T)-\frac{1}{2}\lambda(X^j(T)+c^j)\right)\right].$$

Here the first term represents the expected trading costs in $[0,T]$ and the second term is the cost of exercising the options. The last term describes the liquidation value of $X^j(T)+c^j$ stock shares under infinitely slow liquidation in $[T,\infty)$. Using the stock price dynamics (2.1), (2.2) and $X^j(0)=0$, it can be shown that this equals

$$\sup_{u^j,c^j}\mathbb{E}\left[\int_0^T -u^j(s)\,g(u^j(s))\,\mathrm{d}s - c^jK^j + c^j\left(P(0)+\sigma B(T)-\frac{1}{2}\lambda c^j\right)\right].$$

The cost term $\int_0^T u^j(s)\,g(u^j(s))\,\mathrm{d}s$ is non-negative and the remaining terms do not depend on u^j, so the optimal trading strategy in the stock is $u^j\equiv 0$. This shows that $u^j\equiv 0$ for each $j\leq N$ is a Nash equilibrium. □

At first glance, proposition 5.5 appears that it might contradict Pirrong (2001, p. 221). He states that "replacement of delivery settlement of futures contracts with cash settlement is frequently proposed to reduce the frequency of market manipulation". While his notion of market manipulation refers to market corners and short squeezes (see also Garbade and Silber (1983)), proposition 5.5 shows that this is not always true for manipulation strategies in the sense of 'punching the close'. It is not beneficial to drive up the stock price at maturity if the option is settled physically and the investor needs to liquidate the stocks he receives at maturity. Any price increase is outweighed by subsequent liquidation and has no positive effect, but it is costly. This confirms a claim made by Kumar and Seppi (1992, p. 1497), who argue that whether "futures contracts with a 'physical delivery' option [are] also susceptible to liquidity-driven manipulation [...] depends on whether 'offsetting' trades can be used to unwind a futures position with little price impact".

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Appendix A: An existence result

In this appendix, we prove theorems 3.1 and 4.1 where the PDE (3.5) in the risk-neutral setting is a special case of the system (4.1) for risk-averse agents, with $\alpha^j = 0$ for each j. In order to establish our existence and uniqueness of equilibrium result, we adopt the proof of proposition 15.1.1 of Taylor (1997) to our framework. After time inversion from t to $T-t$, both systems of PDEs are of the form

$$v_t = Lv + F(v_p), \tag{A1}$$

for $v \triangleq (v^1, \ldots, v^N)$, where L is the Laplace operator

$$L = \frac{1}{2}\sigma^2 \frac{\partial^2}{\partial p^2},$$

and $F = (F^1, \ldots, F^N)$ is of the form

$$F^j(v_p) = -\frac{1}{2}\sigma^2 \alpha^j (v_p^j)^2 + \lambda\left(u^* - \frac{g(u^*)}{g'(u^*)}\right) v_p^j + \frac{g(u^*)^2}{g'(u^*)}.$$

Here $u^* = u^*(v_p)$ is given implicitly by (3.3). The initial condition is

$$v(0, p) = H(p) = (H^1, \ldots, H^N). \tag{A2}$$

We rewrite (A1) in terms of an integral equation as

$$v(t) = e^{tL} + \int_0^t e^{(t-s)L} F(v_p(s))\, \mathrm{d}s \triangleq \Gamma v(t), \tag{A3}$$

and seek a fixed point of the operator Γ on the following set of functions:

$$\mathbb{X} = \mathcal{C}_b^1(\mathbb{R}, \mathbb{R}^N) \triangleq \{v \in \mathcal{C}^1(\mathbb{R}, \mathbb{R}^N) \mid v, v_p \text{ bounded}\},$$

equipped with the norm

$$\|v\|_{\mathbb{X}} \triangleq \|v\|_\infty + \|v_p\|_\infty.$$

We set $\mathbb{Y} \triangleq \mathcal{C}_b$. Note that $\mathbb{X}$ and $\mathbb{Y}$ are Banach spaces and the semi-group e^{tL} associated with the Laplace operator is strongly continuous on $\mathbb{X}$, sends $\mathbb{Y}$ on $\mathbb{X}$ and satisfies

$$\|e^{tL}\|_{\mathcal{L}(\mathbb{Y}, \mathbb{X})} \le Ct^{-\gamma},$$

for some $C > 0$, $\gamma < 1$ and $t \le 1$. Furthermore, the nonlinearity F is locally Lipschitz and belongs to $\mathcal{C}^\infty$. Indeed, if we apply the implicit function theorem to u^* given by (3.3), we see that the map $a \mapsto u^*(a)$ is $\mathcal{C}^\infty$ with first derivative

$$\frac{\partial}{\partial v_p} u^*(v_p) = \frac{\lambda}{(N+1)\, g'(u^*(v_p)) + u^*(v_p)\, g''(u^*(v_p))},$$

where the denominator is positive due to assumption 2.1. The cost function g is $\mathcal{C}^\infty$ by assumption. In particular, the assumptions of proposition 15.1.1 of Taylor (1997) are satisfied.

Before we proceed, we need the following lemma. It states that the value function satisfies $\|V^j\|_{\mathbb{X}} \le K$ for each $j \le N$ and some constant K, so it suffices to construct a solution in the following set:

$$\mathbb{X}_K \triangleq \{v \in \mathbb{X} \mid \|v\|_{\mathbb{X}} \le K\}.$$

Lemma A.1: *There is a constant K such that $\|V^j\|_{\mathbb{X}} \le K$ for each $j \le N$.*

Proof: We prove the assertion for risk-neutral agents, and the CARA case follows by the same arguments. Our *a priori* estimates of proposition 2.5 yield that the trading strategy u^j is bounded for each $j \le N$, and hence the aggregate trading strategy u^* is also bounded. By definition, the value function $V^j(t, p)$ is then also bounded. Finally, equation (3.4) implies that v_p^j is bounded. □

We are now ready to prove existence and uniqueness of a solution to (A3). In a nutshell, the argument is the following. Using proposition 15.1.1 of Taylor (1997), we construct a solution to (A1)–(A2) for a small time horizon $[0, \tau]$, with $\tau > 0$ specified below. The vector v is the vector of value functions by theorem IV.3.1 of Fleming and Soner (1993), so by lemma A.1 the constructed solution is in $\mathbb{X}_K$. We apply this argument recursively to extend the solution to $[0, T]$.

Proposition A.2: *There is $\tau > 0$ such that, for each $n \in \mathbb{N}_0$, the PDE* (A3) *with initial condition* (A2) *admits a unique classical, bounded solution in $\mathbb{X}_K$ on the time horizon* $[0, n\tau \wedge T]$. This solution is the value function.

Proof:

(1) For $n = 0$, there is nothing to prove. Pick $n \in \mathbb{N}$ such that $n\tau < T$. By induction, we can assume that there is a solution $v^{(n)} \in \mathbb{X}_K$ on the time horizon $[0, n\tau]$. In particular, the initial condition for the next recursion step $h^{(n)} \triangleq v^{(n)}(n\tau)$ is in $\mathbb{X}_K$.

(2) Fix $\delta > 0$. We construct a short time solution on the following set of functions:

$$Z^{(n+1)} \triangleq \{v \in \mathcal{C}([n\tau, (n+1)\tau], \mathbb{X}) \mid v(n\tau) = h^{(n)}, \|v(t) - h^{(n)}\|_{\mathbb{X}} \leq \delta \ \forall t \in [n\tau, (n+1)\tau]\}.$$

We first show that Γ: $Z^{(n+1)} \to Z^{(n+1)}$ is a contraction, if $\tau > 0$ is chosen small enough. For this, let τ_1 be small enough such that, for $t \leq \tau_1$ and any $v \in \mathbb{X}_K$, we have

$$\|e^{tL} v - v\|_{\mathbb{X}} \leq \frac{1}{2}\delta.$$

Here we have used that e^{tL} is a continuous semigroup and $\|v\|_{\mathbb{X}} \leq K$. In particular, for $v = h^{(n)}$,

$$\|e^{tL} h^{(n)} - h^{(n)}\|_{\mathbb{X}} \leq \frac{1}{2}\delta.$$

For $v \in Z^{(n+1)}$, the derivative v_p is uniformly bounded in the sense $\|v_p\|_\infty \leq \|h^{(n)}\|_{\mathbb{X}} + \delta \leq K + \delta$. Hence, we only evaluate F on compact sets. By assumption, F is locally Lipschitz. In particular, F is Lipschitz on compact sets. In other words, there is a constant K_1 such that, for any v, $w \in Z^{(n+1)}$, we have

$$\|F(v_p) - F(w_p)\|_{\mathbb{Y}} \leq K_1 \|v - w\|_{\mathbb{X}}.$$

This implies, for $w = h^{(n)}$,

$$\|F(v_p)\|_{\mathbb{Y}} \leq \|F(h_p^{(n)})\|_{\mathbb{Y}} + K_1 \|v - h^{(n)}\|_{\mathbb{X}} \leq K + K_1 \delta \triangleq K_2.$$

This, together with the boundedness assumption on e^{tL}, yields

$$\left\| \int_{n\tau}^{t} e^{(t-y)L} F(v_p(y))\, \mathrm{d}y \right\|_{\mathbb{X}} \leq t\|e^{tL}\| \sup_{n\tau \leq y \leq t} \|F(v_p(y))\|_{\mathbb{Y}} \leq t^{1-\gamma} C K_2.$$

This quantity is $\leq \frac{1}{2}\delta$ if $t \leq \tau_2 \triangleq (\delta / 2CK_2)^{1/(1-\gamma)}$.

Finally, it follows that, for $v \in Z^{(n+1)}$, we have

$$\|\Gamma v - h^{(n)}\|_{\mathbb{X}} \leq \|e^{tL} h^{(n)} - h^{(n)}\|_{\mathbb{X}} + \left\| \int_{n\tau}^{t} e^{(t-y)L} F(v_p(y))\, \mathrm{d}y \right\|_{\mathbb{X}} \leq \frac{1}{2}\delta + \frac{1}{2}\delta = \delta.$$

This shows that Γ maps $Z^{(n+1)}$ into itself.

It remains to show that Γ is a contraction. Let v, $w \in Z^{(n+1)}$. Then

$$\begin{aligned}\|\Gamma v(t) - \Gamma w(t)\|_{\mathbb{X}} &= \left\| \int_{n\tau}^{t} e^{(t-y)L} [F(v_p(y)) - F(w_p(y))]\, \mathrm{d}y \right\|_{\mathbb{X}} \\ &\leq t\|e^{tL}\| \sup_{n\tau \leq y \leq t} \|F(v_p(y)) - F(w_p(y))\|_{\mathbb{Y}} \\ &\leq t^{1-\gamma} C K_2 \sup_{n\tau \leq y \leq t} \|v(y) - w(y)\|_{\mathbb{X}}.\end{aligned}$$

The quantity $t^{1-\gamma} C K_2$ is $\leq \frac{1}{2}$ if $t \leq \tau_3 \triangleq (1/2CK_2)^{1/(1-\gamma)}$. This proves that Γ is a contraction in $Z^{(n+1)}$, if τ is small in the sense

$$0 < \tau \triangleq \min\{\tau_1, \tau_2, \tau_3\}.$$

Note that the time step τ does not depend on n. It is the same in every recursion step.

(3) It follows that Γ has a unique fix point v in $Z^{(n+1)}$. In other words, we constructed a function $v \in \mathcal{C}([n\tau, (n+1)\tau], \mathbb{X}) = \mathcal{C}^{0,1}\ [n\tau, (n+1)\tau]$ that solves the PDE (A3) with initial condition $v(s) = h^{(n)} = v^{(n)}(n\tau)$ on the time interval $[n\tau, (n+1)\tau]$. This solution is actually in $\mathcal{C}^{1,2}((n\tau, (n+1)\tau] \times \mathbb{R}, \mathbb{R}^N)$, due to proposition 15.1.2 of Taylor (1997). Furthermore, v is bounded by construction. Indeed, $\|v\|_\infty \leq \|h^{(n)}\|_{\mathbb{X}} + \delta \leq K + \delta$. We define the new solution as

$$v^{(n+1)} \triangleq v^{(n)} \mathbb{1}_{\{0 \leq t \leq n\tau\}} + v \mathbb{1}_{\{n\tau < t \leq (n+1)\tau\}}.$$

By construction, $v^{(n+1)}$ solves (A3) on the time horizon $[0, (n+1)\tau]$ and is bounded and in $\mathcal{C}^{1,2}$. Hence, we can apply the Verification Theorem IV.3.1 from Fleming and Soner (1993), which yields that $v^{(n+1)}$ is the vector of value functions (up to time reversal). Due to lemma A.1 we have $v^{(n+1)} \in \mathbb{X}_K$. In particular, $\|v^{(n+1)}((n+1)\tau)\|_{\mathbb{X}} \leq K$, which is necessary for the next recursion step.

This completes the proof. □

Appendix B: Proof of propositions 5.3 and 5.4

The argument is the same for both propositions. Fix $N \in \mathbb{N}$. The aggregate trading speed for N players is given from equation (3.8) as

$$u^* = \sum_{i=1}^{N} u^i = \frac{\lambda}{\kappa} \frac{1}{N+1} v_p,$$

where the aggregate value function $v=\sum_{i=1}^N v_i$ from (3.9) solves Burgers' equation

$$0=v_t+\frac{1}{2}\sigma^2 v_{pp}+\frac{\lambda^2}{\kappa}\frac{N}{(N+1)^2}v_p^2, \tag{B1}$$

with terminal condition $v(T,p)=\sum_{i=1}^N H^i(p)=H^1(p)\triangleq H(p)$. On the other hand, the aggregate trading speed for $N+1$ players is

$$\bar{u}^*=\sum_{i=1}^{N+1}\bar{u}^i=\frac{\lambda}{\kappa}\frac{1}{N+2}w_p,$$

where the aggregate value function $w=\sum_{i=1}^{N+1} w_i$ solves

$$0=w_t+\frac{1}{2}\sigma^2 w_{pp}+\frac{\lambda^2}{\kappa}\frac{N+1}{(N+2)^2}w_p^2,$$

with terminal condition $w(T,p)=H(p)$. We have to show that $u^*\geq\bar{u}^*$. To this end, let us define

$$\tilde{w}\triangleq\frac{N+1}{(N+2)^2}\frac{(N+1)^2}{N}w.$$

It is sufficient to show that $v_p\geq\tilde{w}_p$, since then

$$\frac{1}{N+1}v_p\geq\frac{1}{N+1}\tilde{w}_p,$$

and, by definition,

$$\frac{1}{N+1}\tilde{w}_p\geq\frac{1}{N+2}w_p.$$

This implies $u^*\geq\bar{u}^*$.

To show $v_p\geq\tilde{w}_p$, first note that $\tilde{w}$ is chosen such that it satisfies the same PDE (B1) as v, namely

$$0=\tilde{w}_t+\frac{1}{2}\sigma^2\tilde{w}_{pp}+\frac{\lambda^2}{\kappa}\frac{N}{(N+1)^2}\tilde{w}_p^2, \tag{B2}$$

with a smaller terminal condition:

$$\tilde{w}(T,p)=\frac{N+1}{(N+2)^2}\frac{(N+1)^2}{N}H(p)\triangleq(1-\delta)H(p).$$

The solutions to (B1) and (B2) are given in lemma C.1 as

$$v(t,p)=c_1\log\int_{\mathbb{R}}\exp(c_2H(c_3z))f_{c_4p,T-t}(z)\,\mathrm{d}z$$

and

$$\tilde{w}(t,p)=c_1\log\int_{\mathbb{R}}\exp(c_2(1-\delta)H(c_3z))f_{c_4p,T-t}(z)\,\mathrm{d}z,$$

with constants c_1, c_2, c_3, $c_4\in\mathbb{R}$ and $\delta\in(0,1)$. To verify $v_p\geq\tilde{w}_p$, it is sufficient to show

$$\frac{\partial}{\partial p}\log\int_{\mathbb{R}}\exp(G)f_{p,1}(z)\,\mathrm{d}z\geq\frac{\partial}{\partial p}\log\int_{\mathbb{R}}\exp((1-\delta)G)f_{p,1}(z)\,\mathrm{d}z,$$

for an increasing function $G\in\mathcal{C}_b^2$. This is equivalent to

$$\frac{\int_{\mathbb{R}}(z-p)e^Gf_{p,1}(z)\,\mathrm{d}z}{\int_{\mathbb{R}}e^Gf_{p,1}(z)\,\mathrm{d}z}\geq\frac{\int_{\mathbb{R}}(z-p)e^{(1-\delta)G}f_{p,1}(z)\,\mathrm{d}z}{\int_{\mathbb{R}}e^{(1-\delta)G}f_{p,1}(z)\,\mathrm{d}z}$$

or

$$\int_{\mathbb{R}}ze^{\delta G}\frac{e^{(1-\delta)G}f_{p,1}(z)\,\mathrm{d}z}{\int_{\mathbb{R}}e^{(1-\delta)G}f_{p,1}(z)\,\mathrm{d}z}\geq\int_{\mathbb{R}}z\frac{e^{(1-\delta)G}f_{p,1}(z)\,\mathrm{d}z}{\int_{\mathbb{R}}e^{(1-\delta)G}f_{p,1}(z)\,\mathrm{d}z}\times\int_{\mathbb{R}}e^{\delta G}\frac{e^{(1-\delta)G}f_{p,1}(z)\,\mathrm{d}z}{\int_{\mathbb{R}}e^{(1-\delta)G}f_{p,1}(z)\,\mathrm{d}z}$$

or

$$\mathrm{cov}_{\mathbb{Q}}(id,e^{\delta G})\geq 0,$$

under the measure $\mathbb{Q}$ with

$$\mathrm{d}\mathbb{Q}\triangleq\frac{e^{(1-\delta)G}f_{p,1}(z)\,\mathrm{d}z}{\int_{\mathbb{R}}e^{(1-\delta)G}f_{p,1}(z)\,\mathrm{d}z}.$$

The covariance of two increasing functions is surely non-negative. This finally proves the assertion $u^*\geq\bar{u}^*$.

It remains to show $\lim_{N\to\infty}\sum_{i=1}^N u^i(t)=0$. We have

$$\begin{aligned}u^*(t,p)&=\sum_{i=1}^N u^i(t)=\frac{\lambda}{\kappa}\frac{1}{N+1}v_p(t,p)\\&=\frac{\partial}{\partial p}\frac{\lambda}{\kappa}\frac{1}{N+1}\frac{\sigma^2\kappa(N+1)^2}{2\lambda^2N}\\&\quad\times\log\int_{\mathbb{R}}\exp\left(\frac{2\lambda^2N}{\sigma^2\kappa(N+1)^2}H(\sigma z)\right)f_{p/\sigma,T-t}(z)\,\mathrm{d}z\\&=\frac{\partial}{\partial p}\frac{\lambda}{\kappa}\frac{1}{N+1}\frac{\sigma^2\kappa(N+1)^2}{2\lambda^2N}\\&\quad\times\log\int_{\mathbb{R}}\exp\left(\frac{2\lambda^2N}{\sigma^2\kappa(N+1)^2}H\left(\sigma z+\frac{p}{\sigma}\right)\right)f_{0,T-t}(z)\,\mathrm{d}z\\&=\frac{\lambda}{\kappa}\frac{1}{N+1}\frac{1}{\sigma}\\&\quad\times\frac{\left\{\begin{array}{l}\int_{\mathbb{R}}H_p(\sigma z+(p/\sigma))\exp([2\lambda^2N/\sigma^2\kappa(N+1)^2]\\ H(\sigma z+(p/\sigma)))f_{0,T-t}(z)\,\mathrm{d}z\end{array}\right\}}{\left\{\begin{array}{l}\int_{\mathbb{R}}\exp([2\lambda^2N/\sigma^2\kappa(N+1)^2]H(\sigma z\\ +(p/\sigma)))f_{0,T-t}(z)\,\mathrm{d}z\end{array}\right\}},\end{aligned}$$

where we have used lemma C.1 in the second line. This expression is non-negative, since $H_p\geq 0$. Furthermore, we have $\|H_p\|_\infty<\infty$ by assumption. It follows that

$$0\leq\sum_{i=1}^N u^i(t)\leq\frac{\lambda}{\kappa}\frac{1}{N+1}\frac{1}{\sigma}\|H_p\|_\infty\overset{N\to\infty}{\longrightarrow}0.$$

This completes the proof.

Appendix C: Burgers' equation

In the proofs of proposition 3.2 and corollary 4.2 we need the solution to a variant of *Burgers' equation*. Recall our notation

$$f_{\mu,\sigma^2}(z)\triangleq\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{(z-\mu)^2}{2\sigma^2}\right).$$

Lemma C.1: *Let $A\in\mathbb{R}_{>0}$, $B\in\mathbb{R}\backslash\{0\}$ and $G:\mathbb{R}\to\mathbb{R}$ be smooth and bounded. The PDE*

$$0=2v_t+Av_{pp}+Bv_p^2,$$

with terminal condition $v(T, p) = G(p)$ *is solved by*

$$v(t,p) = \frac{A}{B}\log\left[\int_{\mathbb{R}} \exp\left(\frac{B}{A}G\left(\sqrt{A}z\right)\right) f_{p/\sqrt{A},\,T-t}(z)\,\mathrm{d}z\right]. \tag{C1}$$

Proof: We use the linear transformation $v(t,p) \triangleq (A/B)\, w(t, p/\sqrt{A})$ and note that

$$v_t = \frac{A}{B}w_t, \quad v_p = \frac{\sqrt{A}}{B}w_p, \quad v_{pp} = \frac{1}{B}w_{pp}.$$

The PDE under consideration is then equivalent to (after canceling the factor A/B)

$$0 = 2w_t + w_{pp} + w_p^2,$$

with terminal condition $w(T,p) = (B/A)G(\sqrt{A}p)$. Next we apply the transformation $w(t,p) \triangleq \log h(t,p)$, which turns the above PDE into

$$0 = h_t + \frac{1}{2}h_{pp},$$

with terminal condition $h(T,p) = \exp((B/A)G(\sqrt{A}p))$. The solution to this heat equation is

$$h(t,p) = \int_{\mathbb{R}} \exp\left(\frac{B}{A}G\left(\sqrt{A}z\right)\right) f_{p,T-t}(z)\,\mathrm{d}z.$$

This function is well defined since G is assumed to be bounded. Now it becomes clear that $v(t,p) = (A/B) \log h(t, p/\sqrt{A})$ is given by (C1). See also Rosencrans (1972). □