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Stationary equilibria in discounted stochastic games with weakly interacting players

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Abstract

We give sufficient conditions for a non-zero sum discounted stochastic game with compact and convex action spaces and with norm-continuous transition probabilities, but with possibly unbounded state space, to have a Nash equilibrium in homogeneous Markov strategies that depends in a Lipschitz continuous manner on the current state. If the underlying state space is compact this yields the existence of a stationary equilibrium. Stochastic games with weakly interacting players provide a probabilistic framework within which to study strategic behavior in models of non-market interactions.

JEL classification: D81; E32

Keywords: Stochastic games; Stationary equilibria; Local interactions; Non-market interactions

1. Introduction

This paper considers infinite horizon discounted stochastic games with compact and convex action spaces and with norm-continuous transition probabilities. We formulate conditions on the games which guarantee existence of stationary equilibria in pure strategies that depend in a Lipschitz continuous manner on the current state.

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Discounted stochastic games have been introduced by Shapley (1953) as a general model of strategic interaction with symmetric information, and have since been intensively analyzed in both the economic and the mathematical literature. The structure of a stochastic game is similar to that of stochastic dynamic programming. The major difference is that instead of one decision maker maximizing his utility over time, stochastic games involve multiple players controlling the dynamics of some state variable. Since a full characterization of equilibria in stochastic games is typically intractable, one usually tries to prove existence of time-homogeneous equilibria in Markovian strategies. In a Markovian equilibrium the players' actions in every period depend only on the current position of the state variable, and so the dynamics of the state sequence can be described by a homogeneous Markov chain.

For countable state spaces a variety of existence theorems for Markov equilibria have been established by, e.g., Shapley (1953), Fink (1964), and Federgruen (1978). The existence of homogeneous Markov equilibria has also been proved in special cases with uncountable state spaces. For instance, Parthasarathy (1982) considered 2-person games in which the state space is the unit interval and where the agents' strategy sets are finite. This was extended to n players, again each having a finite strategy set, in Parthasarathy and Sinha (1989). Nowak (1985) also worked with an uncountable state space and two players, both of whose action spaces are compact metric spaces. Under fairly general conditions this author showed that such games have an ϵ-equilibrium stationary Markov strategies. Nowak and Raghavan (1992) proved existence of correlated equilibria in stationary strategies. In a correlated equilibrium the behavior of the players is coordinated by a signal transmitted by a fictitious mediator. Under a norm-continuity condition on the transition probabilities Mertens and Parthasarathy (1987) discussed the existence of subgame-perfect, but not necessarily Markovian equilibria in games with uncountable state and action spaces. An alternative proof which is based on selection theorems for measurable correspondences is given in Solan (1998); Chakrabarti (1999) extended the results of Mertens and Parthasarathy (1987) to Markov strategies.

However, no general existence result is yet available. Even less is known about existence of equilibria which display additional continuity properties. The latter issue is of particular interest for games with norm continuous transition rules. In such games the dynamics of the equilibrium process can be described by a Markov chain that has the Feller property if the underlying equilibrium strategy itself depends in a continuous manner on the current state. If, in addition, the game has a compact state space, then the equilibrium is even ergodic. This means that the game admits an initial distribution such that the state sequence is stationary and ergodic. The existence of (correlated) ergodic equilibrium processes has been addressed in the context of *finite-horizon* stochastic games with mutually absolutely continuous transition probabilities by Duffie et al. (1994). These authors give a variety of reasons for focussing on ergodic equilibrium processes. For instance, such equilibria "constitute the simplest sort of equilibria and are thus perhaps focal," and "there is [...] the suspicion that other equilibria require implausible [...] coordination." Guesnerie and Woodford (1992) point out that "an equilibrium that does not display minimal regularity through time—maybe stationarity—is unlikely to generate the coordination between agents that it assumes." Duffie et al. (1994) conclude that "whatever the additional merits of ergodic equilibria are, stationarity is the basis of all econometric models." This calls for general existence results of continuous equilibria in Markovian strategies.

To the best of our knowledge the existence of continuous equilibria has so far only been established in the context of a specific capital accumulation game by Amir (1996) and for supermodular games by Curtat (1996). The latter approach is based on Topkis' (1978) monotonicity theorem. It uses lattice theoretic arguments and relies on complementarity and monotonicity assumptions. Complementarities occur when the marginal utility to one player of undertaking an action is increasing in the number of peers undertaking the same action. This paper provides a different and more unified approach that applies beyond the setting of supermodular games. Instead of imposing monotonicity conditions on the agents' utility functions we consider stochastic games in which the interaction between different players is weak enough. To this end, we first extend the notion of *Moderate Social Influences* introduced by Glaeser and Scheinkman (2000) and enhanced in Horst and Scheinkman (2002) to dynamic games. In a second step we reduce the dynamic decision problem to a static game through the introduction of average continuation functions. This reduction allows us to view an agent's decision problem as an optimization problem depending on some parameters: the actions taken by all the other players and the current position of the state sequence. Montrucchio (1987) gave sufficient conditions for such optimization problems to have optimal solutions that are Lipschitz continuous functions of the parameters. Combining these results with our weak interaction condition, we show that the reduced one shot game has a unique equilibrium that is Lipschitz continuous in the state variable. The key observation is that the Lipschitz constant can be chosen independently of the specific average continuation function. In a third step, we prove existence of Lipschitz continuous equilibria using results from the theory of dynamic programming.

Stochastic games with weakly interacting players are tailor-made to study dynamic microeconomic models of non-market interactions. Non-market interactions are interactions between a large number of agents that are not regulated through a price mechanism. They represent an important aspect of many socio-economic phenomena. For example, the decision of a teen to commit a criminal act or to drop out of high school is often importantly influenced by the related decisions of his friends as documented by Glaeser et al. (1996) and Crane (1991), respectively. Jones (1994) identified smoking habits as another phenomenon where peer group effects play an important role. But social interactions occur not only between peers. They also occur between family members, between ethnic group, and between neighbors in a geographical space. Topa (2001) showed that neighborhood effects are important determinants of employment search; ethnic group effects can explain segregation (Benabou, 1993) and income inequalities (Durlauf, 1992) across cities. Cooper and John (1988) showed that local technological spillover effects are an important determinant of the variation in aggregate output. If production processes are affected by spillovers, small changes in economic fundamentals may be transformed into large changes in aggregate output. Such multiplier effects are a characteristic feature of models of non-market interactions. They provide a possible explanation for the emergence of large fluctuations of aggregate endogenous variables relative to changes in exogenous quantities. But for the multiplier to be well defined, one has to place a quantitative bound on the strength of interactions. Otherwise extreme forms of "herding" may emerge, and the multiplier effects become unbounded. This calls for models of weakly interacting players.

The empirical evidence of peer and neighborhood effects has triggered an increasing theoretical literature studying *static* economies with non-market interactions; see, for instance Glaeser et al. (1996) or Brock and Durlauf (2001). However, the literature on local interactions has not yet been fully integrated into the *dynamic* analysis of equilibrium. When dynamic economies are studied, the analysis is typically confined to the case of backward looking myopic dynamics. Either as a simple explicit dynamic process with random sequential choices as in Brock and Durlauf (2001), or, under a weak interaction condition, as an equilibrium selection procedure for static economies as in, e.g., Glaeser and Scheinkman (2000). One exception is the paper by Bisin et al. (2002). These authors proved the existence of rational expectations equilibria of random economies with locally interacting agents under the assumption that the interaction between different players is not too strong. At the same time they considered an interaction structure that excludes strategic behavior.

The weak interaction approach suggested in this paper provides a unified framework for integrating strategic behavior into dynamic models of social interactions. The framework is flexible enough to allow for both local and global interactions. Local interactions capture situations where agents interact only with a small set of other agents (friends, family members, “neighbors,” etc.) in an otherwise large population. Local interactions are best thought of as being direct. That is, agents’ instantaneous utility functions depend directly on observable choices of neighbors. Interactions are global if people are affected by the average behavior in the population. In a large population the actual average behavior is unlikely to be observable. Instead, it is more natural to assume that agents receive noisy signals about aggregate quantities. Therefore, global interactions are best modeled as indirect interactions. This means that the dependence of payoffs on the average behavior is felt only through the impact of aggregate quantities on the dynamics of the state sequence. Models of local and global interactions allow for a combination of *local* externalities like neighborhood effects with *global* externalities like fashions on which an individual agent in a large population only has a small impact. Our framework also allows us to integrate the standard economic analysis in which interactions are mediated by global quantities like prices, wages or per capita human capital into the analysis of peer and group effects captured by local interactions. As an illustration we consider a model of economic growth where local technological spillovers affect the efficiency of production processes.

The remainder of the paper proceeds as follows. The model and the main results are presented in Section 2. Section 3 illustrates the range of applications of stochastic games with weakly interacting players. In Section 4 the dynamic decision problem is reduced to a static game. Section 5 proves our main results. Section 6 concludes.

2. Lipschitz continuous Nash equilibria in stochastic games

The stochastic games $\Sigma = (I, M, (\underline{\overline{X}}^i, U^i, \beta^i), Q, \xi)$ that we consider in this paper are defined in terms of the following objects:

- The set of *players* is the finite set $I = \{1, 2, \ldots, N\}$.

- The *state space* M is a convex subset of a normed space $(H, \|\cdot\|_M)$. The state space is equipped with its Borel-σ-field $\mathcal{M}$.
- The *action space* $\underline{\overline{X}}^i$ of the player i is a closed, compact and convex subset of some Hilbert space $(H^i, \|\cdot\|_i)$. A typical action of player i is denoted x^i. The actions taken by player i's competitors are denoted $x^{-i} \in \underline{\overline{X}}^{-i} := \{x^{-i} = (x^j)_{j \in I \setminus \{i\}}\}$, and $\underline{\overline{X}} := \{x = (x^i)_{i \in I}\colon x^i \in \underline{\overline{X}}^i\}$ is the compact set of all *action profiles*.
- The *utility function* of player i is a continuous map $U^i : M \times \underline{\overline{X}} \to \mathbb{R}$.
- The *discount factor* of player i is $\beta^i \in (0, 1)$.
- The *law of motion* Q is a stochastic kernel from $M \times \underline{\overline{X}}$ to M.
- The *starting point* of the state sequence is $\xi \in M$.

In reaction to the current state $\xi_t \in M$, the players take their actions $x_t^i = \tau^i(\xi_t)$ independently of each other according to a *Markov strategy* $\tau^i : M \to \underline{\overline{X}}^i$. The restriction to Markovian strategies does not pose any difficulties because any equilibrium when players are restricted to Markovian strategies also constitutes an equilibrium in a game where the players' actions depend on the entire history of the state sequence.

The selected action profile $x_t = (x_t^i)_{i \in I}$ along with the present state ξ_t yields the instantaneous payoff $U^i(\xi_t, x_t) = U^i(\xi_t, x_t^i, x_t^{-i})$ to the agent $i \in I$. The distribution of the new state is $Q(\xi_t, x_t; \cdot)$. An initial distribution ν on M along with a Markov strategy $\tau = (\tau^i)_{i \in I}$ induces a probability measure $\mathbb{P}_\nu^\tau$ on the canonical path space in the usual way. Under $\mathbb{P}_\nu^\tau$ the state sequence is a Markov chain, and the *expected discounted reward* to player $i \in I$ is given by

$$J^i(\xi, \tau) := \mathbb{E}_\nu^\tau \left[\sum_{t=0}^{\infty} \left(\beta^i\right)^t U^i(\xi_t, x_t) \right]. \tag{1}$$

Here the expectation is taken with respect to the measure $\mathbb{P}_\nu^\tau$. As usual, a Markov strategy τ will be called a *Nash equilibrium* if no player can increase his payoff by unilateral deviation:

$$J^i(\xi, \tau) = J^i\left(\xi, \tau^i, \tau^{-i}\right) \geqslant J^i\left(\xi, \sigma^i, \tau^{-i}\right) \quad \text{for all } \sigma^i : M \to \underline{\overline{X}}^i \text{ and each } i \in I. \tag{2}$$

Henceforth, a Nash equilibrium in Markovian strategies τ will simply be called an equilibrium. We say that τ is Lipschitz continuous, if there exists a finite constant L^* such that

$$\left\| \tau^i(\xi) - \tau^i\left(\hat{\xi}\right) \right\|_M \leqslant L^* \left\| \xi - \hat{\xi} \right\|_M \quad \text{for each } i \in I \text{ and all } \xi, \hat{\xi} \in M.$$

This paper gives conditions that guarantee existence of Lipschitz continuous equilibria. In a first step, we impose continuity conditions on the utility functions and the law of motion.

Assumption 2.1. (i) The utility functions are bounded and Lipschitz continuous: There exists a constant $L > 0$ such that

$$\begin{aligned} &\left| U^i(\xi_1, x) - U^i(\xi_2, y) \right| \\ &\quad \leqslant L\left(\|\xi_1 - \xi_2\|_M + \|x - y\| \right) \quad \text{for each } \xi_1, \xi_2 \in M \text{ and } \quad x, y \in \underline{\overline{X}}. \end{aligned}$$

Here $\|x\| := \max_i \|x^i\|_i$ denotes the norm on $\underline{\overline{X}}$.

(ii) For all $(\xi, x) \in M \times \underline{\overline{X}}$, the probability measure $Q(\xi, x; \cdot)$ has a density $q(\xi, x, \cdot)$ with respect to some measure μ on $(M, \mathcal{M})$, i.e.,

$$\mathrm{d}Q(\xi, x; \cdot) = q(\xi, x, \cdot)\,\mathrm{d}\mu.$$

For each $\xi_1, \xi_2 \in M$ and every $x, y \in \underline{\overline{X}}$, the densities satisfy the Lipschitz condition

$$\bigl|q(\xi_1, x, \eta) - q(\xi_2, y, \eta)\bigr| \leqslant L\bigl(\|\xi_1 - \xi_2\|_M + \|x - y\|\bigr). \tag{3}$$

The Lipschitz continuity condition (3) translates into a norm-continuity condition on the transition probabilities $Q(\xi, x; \cdot)$. If $\xi_n \to \xi$ and $x_n \to x$, then

$$\sup_{B \in \mathcal{M}} \bigl|Q(\xi_n, x_n; B) - Q(\xi, x; B)\bigr| \to 0 \quad \text{as } n \to \infty.$$

Norm-continuity conditions have also been imposed by, e.g., Mertens and Parthasarathy (1987) and Duffie et al. (1994). Assumption 2.1 is sufficient to prove existence of equilibria in mixed strategies. In order to prove existence of *continuous* equilibria we will also assume strong concavity of an agent's utility function which respect to his own action. In addition, we need to place a quantitative bound on the strength of interactions between different players. That is, we will assume that both the agents' instantaneous utility functions and the transition densities are only weakly affected by changes in players actions. We formulate our weak interaction condition in terms of a perturbation of the *Moderate Social Influence* assumption introduced in Glaeser and Scheinkman (2000). The following section illustrates the latter condition in a situation where the utilities and the densities are sufficiently smooth.

2.1. Assumptions and the main results; the differentiable case

In this subsection we consider the special case where $M, \underline{\overline{X}}^1, \ldots, \underline{\overline{X}}^N \subset \mathbb{R}$ are compact intervals, and where the utility functions and the densities are at least twice continuously differentiable. We use the notation

$$U^i_{i,j}(\xi, x) := \frac{\partial^2}{\partial x^i \partial x^j} U^i(\xi, x) \quad \text{and} \quad q_{i,j}(\xi, x, \eta) := \frac{\partial^2}{\partial x^i \partial x^j} q(\xi, x, \eta),$$

In order to introduce a weak interaction condition for stochastic games, we fix an initial state ξ, an action profile x, and *average continuation functions* $f^i : M \to \mathbb{R}$. The map $f^i : M \to \mathbb{R}$ specifies the rewards the player i expects to receive from time $t = 2$ on. Thus, his actual expected payoff is

$$V^{i,f}(\xi, x) := U^i(\xi, x) + \beta^i \int f^i(\eta) q(\xi, x, \eta) \mu(\mathrm{d}\eta). \tag{4}$$

Hence we can define the *static* one-shot games

$$\Sigma_{f,\xi} = \bigl(\underline{\overline{X}}^1, \ldots, \underline{\overline{X}}^N, V^{1,f}(\xi, \cdot), \ldots, V^{N,f}(\xi, \cdot)\bigr)$$

with payoff functions $V^{i,f}(\xi, \cdot)$, and with action sets $\underline{\overline{X}}^i$. Following Glaeser and Scheinkman (2000), we say that *Moderate Social Influence* (MSI for short) prevails in $\Sigma_{f,\xi}$ if the marginal utility of an agent's own action is less affected by a change in all the other players' choices than by a change of his own action. Specifically, MSI prevails if there exists $\gamma < 1$ such that

$$\sum_{j \neq i} \sup_x \frac{|V^{i,f}_{i,j}(\xi, x)|}{|V^{i,f}_{i,i}(\xi, x)|} \leqslant \gamma \quad \text{for all } \xi \in M, \text{ and every } i \in I. \tag{5}$$

This weak interaction condition guarantees uniqueness of equilibria in $\Sigma_{f,\xi}$. It also excludes "herding behavior" where, for instance, all players copy the behavior of some "leader." In particular, the MSI condition guarantees that the multiplier effects in $\Sigma_{f,\xi}$ are well defined. This means that a small perturbation of the current state cannot have an unbounded effect on the average behavior throughout the entire set of players; see Glaeser and Scheinkman (2000) or Horst and Scheinkman (2002) for further details.

A standard argument in discounted dynamic programming shows that the game Σ has an equilibrium, if there exist average continuation functions $F^i : M \to \mathbb{R}$ such that, in equilibrium, the one-shot game $\Sigma_{F,\xi}$ satisfies

$$V^{i,F}(\xi, x) = U^i(\xi, x) + \beta^i \int F^i(\eta) q(\xi, x, \eta) \mu(\mathrm{d}\eta) = F^i(\xi) \quad \text{for all } \xi \in M \text{ and } i \in I. \tag{6}$$

Under the *Moderate Social Influence* condition the game $\Sigma_{F,\xi}$ has a unique equilibrium that depends continuously on ξ as shown by Horst and Scheinkman (2002). Thus, if there exists an average continuation function such that (6) holds, and if MSI prevails in the static game $\Sigma_{F,\xi}$, then Σ has a continuous equilibrium. Since the class of average continuation functions can a priori not be restricted except for

$$\|f^i\|_\infty \leqslant \sum_{t=0}^{\infty} (\beta^i)^t \|U^i\|_\infty = \frac{1}{1-\beta^i} \|U^i\|_\infty,$$

it is natural to assume that (5) holds uniformly in all average continuation functions. That is, independently of what a player expects to receive in the future, his marginal utility at time $t = 1$ is always more affected by changes in his own action than by changes in the other agents' choices. In order to make this more precise, we denote by $\|q_{i,j}(\xi, x, \cdot)\|_{L^1} := \int |q_{i,j}(\xi, x, \eta)| \mu(\mathrm{d}\eta)$ the $L^1(\mu)$-norm of the random variable $q_{i,j}(\xi, x, \cdot)$. Since

$$\left|V^{i,f}_{i,j}(\xi_1, x_1)\right| \leqslant \left|U^i_{i,j}(\xi_1, x_1)\right| + \beta^i \|f^i\|_\infty \left\|q_{i,j}(\xi_1, x_1, \cdot)\right\|_{L^1},$$

an extension of the weak interaction condition in Horst and Scheinkman (2002) to dynamic games can be formulated in terms of the following condition.

Assumption 2.2. Let $\beta := \max_i \beta^i$. There exists $\gamma < 1$ such that, for all $i \in I, \xi \in M$,

$$\sum_{j \neq i} \sup_x \frac{|U^i_{i,j}(\xi, x)|}{|U^i_{ii}(\xi, x)|} + \frac{\beta}{1-\beta} \|U^i\|_\infty \sum_{j \in I} \sup_x \frac{\|q_{i,j}(\xi, x, \cdot)\|_{L^1}}{|U^i_{ii}(\xi, x)|} \leqslant \gamma. \tag{7}$$

We are now ready to state a first existence result for Lipschitz continuous equilibria of stochastic games with compact state spaces. The proof is similar to the one of Theorem 2.10 below.

Theorem 2.3. *Let Σ be a stochastic game where $M, \underline{\overline{X}}^1, \ldots, \underline{\overline{X}}^N \subset \mathbb{R}$ are convex and compact. If Assumption* 2.1 *and the Moderate Social Influence Assumption* 2.9 *hold, then Σ has a Lipschitz continuous equilibrium.*

If $\{\xi_t\}$ is an exogenous Markov chain whose dynamics cannot be controlled by the players, then $q_{i,j} \equiv 0$. The same holds if the agents share a common convex action set $\underline{\overline{Y}}$, and if the law of motion takes the form

$$Q(\xi, x; \cdot) = \varrho(x) Q_1(\xi; \cdot) + (1 - \varrho(x))\, Q_2(\xi; \cdot) \quad \text{where} \quad \varrho(x) = \frac{1}{N} \sum_{i \in I} x^i \tag{8}$$

denotes the average action of all players. In both cases Assumption 2.2 reduces to the *diagonal dominance condition*

$$\sum_{j \neq i} \sup_x \frac{|U^i_{i,j}(\xi, x)|}{|U^i_{i,i}(\xi, x)|} \leqslant \gamma < 1 \quad \text{for all } \xi \in M, \text{ and every } i \in I. \tag{9}$$

This is the *Moderate Social Influence* condition in Horst and Scheinkman (2002) for *static* games with payoff functions U^i. If the law of motion depends in a more general manner on the average action taken by all the agents, then the MSI condition translates into a perturbation of the diagonal dominance condition. In situations where the densities take the form $q(\xi, x, \eta) = \varphi(\xi, \varrho(x), \eta)$ for a smooth function $\varphi : M \times \underline{\overline{Y}} \times M \to \mathbb{R}_+$ we have

$$q_{i,j}(\xi, x, \eta) = \frac{1}{N^2} \varphi_{22}\big(\xi, \varrho(x), \eta\big).$$

Thus, there exist constants $C^i < \infty$ such that the *MSI* condition holds if

$$\sum_{j \neq i} \sup_x \frac{|U^i_{i,j}(\xi, x)|}{|U^i_{i,i}(\xi, x)|} + \frac{C^i}{N} \leqslant \gamma < 1 \quad \text{for all } \xi \in M, \text{ and for each } i \in I.$$

If the constants C^i are uniformly bounded, Assumption 2.9 reduces to the diagonal dominance condition (9) for $N \to \infty$. If the densities depend on x through a weighted average of the form $\sum_{i \in I} \zeta^i x^i$, then MSI prevails if the utility functions satisfy (9) and if the constants ζ^i are small enough.

Remark 2.4. Loosely speaking, the result formulated in Theorem 2.3 may be interpreted as saying that if a game a close to being anonymous (see, e.g., the seminal paper by Jovanovic and Rosenthal (1988) for a detailed analysis of anonymous games), then an equilibrium exists.

Stochastic games with weakly interacting actions are tailor-made to analyze dynamic games of non-market interactions. Non-market interactions are interactions between many players that are not regulated through a price mechanism. Games of non-market interactions will be studied in Section 3. We close this subsection with a first example where our MSI condition can easily be verified.

Example 2.5. Assume that the agents' action sets are $\underline{X}^i = [-1, 1]$. Assume also that the law of motion takes the linear form (8), and that the utility functions are given by

$$U^i(\xi, x) = -\frac{J}{2}\left(x^i - \frac{1}{N-1}\sum_{j\neq i} x^j\right)^2 - \frac{1-J}{2}\left(x^i - \xi\right)^2 + \theta^i x_t^i. \tag{10}$$

Utility functions of the form (10) are standard in the literature on non-market interactions; see, e.g., Brock and Durlauf (2001), Glaeser and Scheinkman (1999) or Glaeser et al. (1996). They capture situations where agents have a desire for conformity. That is, they capture situations where the agents prefer to take the same actions as their peers. The taste for conformity is measured by the parameter $J \in (0, 1)$. The quantity θ^i may be viewed as an individual parameter that specifies the agent's type. The MSI condition is satisfied since $q_{i,j} \equiv 0$, because $U^i_{i,j} = -J/(N-1)$ and because $U^i_{i,i} = 1$. Thus, the game has a Lipschitz continuous equilibrium. Since quadratic utility functions are not monotone in neighbors choices, the game is not supermodular. Therefore, existence of continuous equilibria cannot be deduced from the results in Curtat (1996).

2.2. Assumptions and the main results: the non-differentiable case

Before we consider games with more general state and action spaces, we recall that a function $f : Y \to \mathbb{R}$ defined on a convex subset Y of some Hilbert space H is called α-concave for $\alpha > 0$, if the map $y \mapsto f(y) + \frac{1}{2}\alpha\|y\|^2$ is concave on Y. We also recall that $f : Y \to \mathbb{R}$ is differentiable at $y \in Y$ in the feasible direction $h \in H$, if $y + th \in Y$ for some $t > 0$, and if the limit $f'(y; h) := \lim_{t\downarrow 0} \frac{1}{t}(f(y + th) - f(y))$ exists and is finite.[1]

Assumption 2.6. (i) There exist $\alpha > 0$ and functions $\alpha^i : M \to (\alpha, \infty)$ such that, for all $x^{-i} \in \underline{X}^{-i}$, the map $U^i(\xi, \cdot, x^{-i})$ is $\alpha^i(\xi)$-concave on $\underline{X}^i$.

(ii) The partial derivatives $U^i_1(\xi, x; h^i)$ of U^i in the coordinate x^i at (ξ, x) exist in all feasible directions $h^i \in H^i$, and the players' marginal utilities are Lipschitz continuous: There exist constants $L^{i,j}(\xi)$ such that

$$\left|U^i_1\left(\xi, x^i, x^{-i}; h^i\right) - U^i_1\left(\xi, x^i, y^{-i}; h^i\right)\right| \leqslant L^{i,j}(\xi)\left\|x^j - y^j\right\|_j \left\|h^i\right\|_i$$

for all actions profiles $x^{-i}, y^{-i} \in \underline{X}^{-i}$ with $x^k = y^k$ for $k \notin \{i, j\}$. Moreover, there are constants L^i such that

$$\left|U^i_1\left(\xi_1, x^i, x^{-i}; h^i\right) - U^i_1\left(\xi_2, x^i, x^{-i}; h^i\right)\right| \leqslant L^i \|\xi_1 - \xi_2\|_M \left\|h^i\right\|_i$$

for all $\xi_1, \xi_2 \in M$ and each $x \in \underline{X}$.

The quantity $L^{i,j}(\xi)$ measures the dependence of agent i's marginal utility on the changes of the choice of player j if the current state is ξ. By analogy, L^i measures the dependence of his marginal utility on the current position of the state sequence.

[1] The connection between α-concavity and differentiability is discussed in Appendix A.

Remark 2.7. We assume strict concavity of an agent's utility function with respect to his own action. Therefore, our model cannot be used to study games with finitely many actions, by defining an auxiliary game with compact action sets in which the set of pure actions coincides with the class of mixed actions in the original game.

We also need to bound the impact of an individual player on the law of motion.

Assumption 2.8. (i) The directional derivative $q_i(\xi, x, \eta; h^i)$ of the density q at (ξ, x, η) in the feasible direction $h^i \in H^i$ exists and $|q_i(\xi, x, \eta; h^i)| \leqslant \varphi(\eta)\|h^i\|_i$ for some $\varphi \in L^1(\mu)$.

(ii) The directional derivatives $q_i(\xi, x, \eta; h^i)$ are Lipschitz continuous. Specifically, there are μ-integrable function $\widehat{L}^{i,j}(\xi, \cdot): M \to \mathbb{R}$ which satisfy

$$\left|q_i\left(\xi, x^i, x^{-i}, \eta; h^i\right) - q_i\left(\xi, x^i, y^{-i}, \eta; h^i\right)\right| \leqslant \widehat{L}^{i,j}(\xi, \eta)\left\|x^j - y^j\right\|_j \left\|h^i\right\|_i \tag{11}$$

for every $\xi \in M$ and all action profiles x^{-i}, y^{-i} with $x^k = y^k$ for all $k \notin \{i, j\}$, and

$$\left|q_i\left(\xi, x^i, x^{-i}, \eta; x^i - \hat{x}^i\right) - q_i\left(\xi, \hat{x}^i, x^{-i}, \eta; x^i - \hat{x}^i\right)\right| \leqslant \widehat{L}^{i,i}(\xi, \eta)\left\|x^i - \hat{x}^i\right\|_i^2.$$

Moreover, there are constants $\widehat{L}^i$ such that

$$\left|q_i\left(\xi_1, x, \eta; h^i\right) - q_i\left(\xi_2, x, \eta; h^i\right)\right| \leqslant \widehat{L}^i \|\xi_1 - \xi_2\|_M \left\|h^i\right\|_i$$

for each $\xi_1, \xi_2 \in M$ and all $x = (x^i, x^{-i}) \in \underline{X}$.

We are now ready to formulate our weak interaction condition in the more general situation where the utility functions and the densities are not twice continuously differentiable. As in the preceding section, we assume that an agent's marginal utility is less affected by a change in his own action than by changes in the other players' choices.

Assumption 2.9. Let $\beta := \max_i \beta^i$, and $\widehat{L}^{i,j}(\xi) := \|\widehat{L}^{i,j}(\xi, \cdot)\|_{L^1}$. There is $\gamma < 1$ such that

$$\sum_{j \neq i} L^{i,j}(\xi) + \frac{\beta}{1-\beta}\left\|U^i\right\|_\infty \sum_{j \in I} \widehat{L}^{i,j}(\xi) \leqslant \gamma \alpha^i(\xi) \tag{12}$$

holds for all $i \in I$ and each $\xi \in M$.

Let us now formulate an extension of Theorem 2.3 that applies to the case of non-smooth utility functions. Its proof will be given in Section 5 below.

Theorem 2.10. *Suppose that the discounted stochastic game Σ has a compact state space M and that Assumption* 2.1 *and Assumptions* 2.6–2.9 *are satisfied. Then Σ has a Lipschitz continuous equilibrium. The Lipschitz constant depends on $\alpha > 0$.*

Following Duffie et al. (1994), we call an equilibrium τ *ergodic* if there exists an initial distribution μ^* such that the state sequence is stationary and ergodic[2] under $\mathbb{P}^\tau_{\mu^*}$. If Σ satisfies the assumptions of Theorem 2.10, then it admits a Lipschitz continuous equilibrium τ.

[2] A Markov chain $\{\xi_t\}$ with state space M is called ergodic under a measure $\mathbb{P}$ if $\lim_{T\to\infty} \frac{1}{T}\sum_{t=1}^T f(\xi_t) = \int f \, d\mathbb{P}$ holds $\mathbb{P}$-a.s. for every bounded measurable function $f: M \to \mathbb{R}$.

The transition operator K^τ of the equilibrium process $\{\xi_t\}$ acts on bounded measurable functions $f : M \to \mathbb{R}$ according to

$$K^\tau f(\cdot) := \int_M f(\eta) K^\tau(\cdot;\, \mathrm{d}\eta) = \int_M f(\eta) Q\big(\cdot, \tau(\cdot);\, \mathrm{d}\eta\big).$$

Since both the densities and the equilibrium strategies are Lipschitz continuous,

$$\begin{aligned}\lim_{n\to\infty}\big|K^\tau f(\xi_n) - K^\tau f(\xi)\big| &\leqslant \lim_{n\to\infty} \|f\|_\infty \big\|q\big(\xi_n, \tau(\xi_n), \eta\big) - q\big(\xi, \tau(\xi), \eta\big)\big\|_\infty \\ &\leqslant \lim_{n\to\infty} L\big(\|\xi_n - \xi\|_M + \big\|\tau(\xi_n) - \tau(\xi)\big\|\big) = 0\end{aligned}$$

if $\lim_{n\to\infty} \xi_n = \xi$. In particular, the Markov chain $\{\xi_t\}$ has the Feller property. This means that the transition kernel K^τ maps the class of all continuous functions $f : M \to \mathbb{R}$ into itself. It is well known (Breiman, 1968) that Feller processes on compact state spaces admit an ergodic invariant distribution. That is, there exists an initial distribution μ^* such that the state sequence is stationary and ergodic under $\mathbb{P}^\tau_{\mu^*}$. If, in addition, the densities are strictly positive, then the Markov chain has at most one invariant measure. In this case the sequence $\{\xi_t\}$ converges in distribution to μ^*, independently of the initial state. Thus, we have the following corollary to Theorem 2.10.

Corollary 2.11. *Under the assumptions of Theorem* 2.10 *the game Σ has an ergodic equilibrium. If, in addition,* $q(\xi, x, \cdot) > 0$*, then the state sequence converges in distribution to* μ^**, independently of the initial condition.*

Theorem 2.10 is applicable to stochastic games with compact, and hence *bounded* state spaces. An extension to games with unbounded state spaces can be established under a mild additional assumption on the densities $q(\xi, x, \cdot)$. To this end, we denote by $M_n \uparrow M \subset H$ an increasing sequence of closed, compact convex sets, and by $q_n : M_n \times \underline{\overline{X}} \times M_n : \to \mathbb{R}$ ($n \in \mathbb{N}$) a sequence of densities with respect to μ which converges to $q(\xi, x, \cdot)$ uniformly on compact sets:

$$\sup_{\eta\in K}\big|q_n(\xi, x, \eta) - q(\xi, x, \eta)\big| \xrightarrow{n\to\infty} 0 \quad \text{for all compact sets } K \subset M. \tag{13}$$

Remark 2.12. Let Q_n be the stochastic kernel from $M_n \times \underline{\overline{X}}$ to M_n that is defined in terms of the densities q_n, and consider the stochastic game $\Sigma_n = (I, M_n, (U^i, \underline{\overline{X}}^i, \beta^i), Q_n, \xi)$. Our condition (13) translates into an assumption on the conditional transition dynamics of the state sequences $\{\xi^n_t\}$ and $\{\xi_t\}$ associated to the respective games Σ_n and Σ. In order to see this, we fix a state $\xi \in M_n$ and an action profile $x \in \underline{\overline{X}}$, and introduce the measures $\mu_n(\xi, x; \cdot)$ and $\mu(\xi, x; \cdot)$ by

$$\mathrm{d}\mu_n(\xi, x; \cdot) = q_n(\xi, x; \cdot)\,\mathrm{d}\mu \quad \text{and} \quad \mathrm{d}\mu(\xi, x; \cdot) = q(\xi, x; \cdot)\,\mathrm{d}\mu, \tag{14}$$

respectively. For any bounded function $h : H \to \mathbb{R}$ with compact support $K \subset H$ we have

$$\lim_{n\to\infty}\bigg|\int_K h(\eta)\big[\mu_n(\xi, x; \mathrm{d}\eta) - \mu(\xi, x; \mathrm{d}\eta)\big]\bigg|$$

$$\leqslant \|h\|_\infty \sup_{\eta \in K} \bigl| q_n(\xi, x, \eta) - q(\xi, x, \eta) \bigr| = 0.$$

Thus, under (13) the sequence $\{\mu_n(\xi, x; \cdot)\}$ converges weakly to $\mu(\xi, x; \cdot)$.

We are now ready to formulate an extension of Theorem 2.10 to stochastic games with unbounded state spaces which will be proved in Section 5 below.

Corollary 2.13. *Let $\Sigma = (I, M, (\underline{\overline{X}}^i, U^i, \beta^i), Q, \xi)$ be a discounted non-cooperative stochastic game. Let $M_n \uparrow M \subset H$ be an increasing sequence of closed compact convex sets, and let $q_n : M_n \times \underline{\overline{X}} \times M_n \to \mathbb{R}$ be densities with respect to some measure μ on $(M, \mathcal{M})$ that satisfy* (13)*. If there exists $\gamma^* < 1$ such that all the games $\Sigma^n = (I, M_n, (\underline{\overline{X}}^i, U^i, \beta^i), Q_n, \xi)$ satisfy the MSI condition* (12) *with $\gamma = \gamma^*$, then Σ has a Lipschitz continuous equilibrium.*

3. Applications of stochastic games with weak interactions

We are now going to illustrate the range of applications of stochastic games with weakly interacting players. Our focus will be on games of non-market interactions, i.e., on strategic interactions between a large number of agents that are not mediated through markets.

3.1. Equilibria in dynamic models of non-market interactions

In this section we develop a dynamic extension of the model of non-market interactions in Glaeser and Scheinkman (2000); see also Horst and Scheinkman (2002). We allow for both *local* and *global* components in the interaction between different players. Social interactions are local if each player interacts only with a small set of other agents in an otherwise large population. Local interactions typically occur between friends or family members. Interactions are global if players are affected by the average behavior throughout the whole population. We assume that a player's instantaneous utility function depends on the choices of others only through his own action and through the observable actions of his neighbors. This captures the idea that observable choices of, e.g., family members have a direct and possibly more distinctive impact on agents' utilities than the average action of all players. On the other hand, in a game with many players, it is unlikely that the average behavior in period t is observable, too. It is more natural to assume that the players only observe signals about $\varrho(x_t)$. This idea will be captured by the fact that the impact of the process $\{\varrho(x_t)\}$ on payoffs is only felt indirectly through its impact on the dynamics of the state sequence.

Let us now be more specific about the structure of the model. Players are infinitely-lived. To each player $i \in I$ we associate his peer or *reference group* $N(i) \subset I \setminus \{i\}$. An agent's peer group may be viewed as the set of players whose actions the agent can actually observe. In large populations, reference groups should thus be thought of as being small relative to the whole set of all players.

In every period t, each player i is subject to a random *taste shock* θ_t^i. The random variables θ_t^i take values in some compact set $\Theta \subset \mathbb{R}^r$. In reaction to his current *type* θ_t^i, the agent i takes an action $x_t^i = \tau^i(\theta_t^i)$ from a common compact and convex action set $\underline{\overline{Y}}$. As in Glaeser and Scheinkman (2000), an agent's instantaneous utility in period t depends on the choices of all the other agents only through a weighted average of the actions chosen by the players in his reference group. To this end, we fix weight factors $\zeta_j^i \geqslant 0$ $(i, j \in I)$ that satisfy $\zeta_j^i = 0$ for $j \notin N(i)$ and $\sum_{j \in N(i)} \zeta_j^i = 1$ for all $i \in I$. Specifically, we assume that preferences at time t are described by a smooth utility function of the form

$$U^i\big(\theta_t^i, x_t\big) = u\big(\theta_t^i, x_t^i, \varrho^i(x_t)\big) \quad \text{where} \quad \varrho^i(x_t) := \sum_{j \in I} \zeta_j^i x_t^j$$

denotes the average choice of player i's peers. The map u is α-concave in its second argument. In our model all heterogeneity across agents is incorporated into neighborhood effects and types. Conditioned on the choices of all agents, the dynamics of the types is described in terms of N independent Markov chains. More precisely, the law of the random variable $\theta_{t+1} = (\theta_{t+1}^i)_{i \in I}$ depends on the current action profile x_t only through the average behavior $\varrho(x_t) = \frac{1}{N} \sum_{i \in I} x^i$. Such an interaction structure captures situations where agents' preferences depend on the unobservable average behavior of all the people only through privately observed signals. Specifically, we assume that the law of motion takes the product form

$$Q(\theta, x; \cdot) := \prod_{i \in I} \pi\big(\theta^i, \varrho(x); \cdot\big) \quad \text{where} \quad \mathrm{d}\pi\big(\theta^i, y; \cdot\big) = \varphi\big(\theta^i, y, \cdot\big)\, \mathrm{d}\lambda. \tag{15}$$

Here $\varphi : \Theta \times \underline{\overline{Y}} \times \Theta \to \mathbb{R}_+$ is smooth, and λ denotes the Lebesgue measure on Θ. An inspection of the proof to Lemma 4.1 shows that for stochastic kernels of the form (15), the quantities $\|q_{i,j}(\xi, x, \eta)\|_{L^1}$ in Assumption 2.9 can be replaced by

$$\left\| \frac{\partial^2}{\partial x^i \partial x^j} \varphi\big(\theta^i, \varrho(x), \cdot\big) \right\|_{L^1} = \frac{1}{N^2} \big\| \varphi_{22}\big(\theta^i, \varrho(x), \cdot\big) \big\|_{L^1}.$$

Observe now that $U_{i,j}^i(\theta^i, x) = \zeta_j^i u_{2,3}(\theta^i, x^i, \varrho^i(x))$ for $i \neq j$. Therefore, *Moderate Social Influence* prevails if there exists $\gamma < 1$ such that, for all $\theta^i \in \Theta$, and each $i \in I$,

$$\sup_{x,i} \frac{|u_{2,3}(\theta^i, x^i, \varrho^i(x))|}{|u_{2,2}(\theta^i, x^i, \varrho^i(x))|} + \frac{\beta}{1-\beta} \sup_{x,i} \frac{\|u\|_\infty}{|u_{ii}(\theta^i, x^i, \varrho^i(x))|} \frac{\|\varphi_{2,2}\|_\infty}{N} \leqslant \gamma.$$

Thus, if the utility function u satisfies a diagonal dominance condition, the game has a Lipschitz continuous equilibrium if the population is large enough. We further illustrate this by means of the following example where preferences are subject to both peer group effects and fashions.

Example 3.1. There are two consumption goods, say good A and good B. A priori, the goods are close substitutes. In each period the agents have to decide which fraction $x_t^i \in [0, 1]$ of their budget for these goods to spend for good A. Personal preferences for good A are described by random variables $\theta_t^i \in [0, 1]$. But the players also have a taste for conformity. They derive utility from consuming the same good as their peers. Such a behavior can frequently be observed among teenagers. For teenagers, brand-name articles often play an important role in identifying themselves as members of certain youth groups.

Specifically, let $N(i) := \{i-1, i+1\}$ where we apply modulo-N arithmetic, and assume that preferences are described by the quadratic utility functions

$$U^i(\theta^i, x) = -\frac{J_1}{2}\left(x^i - \frac{x^{i-1} + x^{i+1}}{2}\right)^2 - \frac{J_2}{2}\left(x^i - \eta^i\right)^2 + \theta_t^i x_t^i.$$

The constants $0 < J_1 < J_2$ satisfy $J_1 + J_2 \leqslant 1$. They measure the taste for conformity. The quantity $\eta_t^i \in [0,1]$ specifies agent i's subjective perception of the average behavior of other players. This reflects the idea that preferences do not only depend on the tastes of peers, but also on fashions. Fashions, in turn, reflect the aggregate behavior through the entire population. For simplicity, we assume that

$$\left(\theta_{t+1}^i, \eta_{t+1}^i\right) \sim \pi\left(\theta_t^i, \eta_t^i, \varrho(x_t); \cdot\right) := Q_1\left(\theta_t^i; \cdot\right) \otimes Q_2\left(\varrho(x_t), \cdot\right)$$

where Q_1 and Q_2 are suitable stochastic kernels. That is, individual types evolve independently of each other in a Markovian manner, and each agent receives a private signal about mean actions. In particular, the law of motion takes the product form (15). The maps U^i are Lipschitz continuous and α-concave with $\alpha = J_1 + J_2$. Moreover, $\|U_{i,i-1}^i\|_\infty = \|U_{i,i-1}^i\|_\infty = J_1$, and $\|U^i\|_\infty = 1$. Thus, our weak interaction condition holds if

$$J_1 + \frac{\beta}{1-\beta}\frac{\|\varphi_{2,2}\|_\infty}{N} < J_2.$$

Thus, the game has a Lipschitz continuous equilibrium if the number of players is large enough and/or if the relative impact of a neighbor's action is weak enough, i.e., if J_2 is big enough.

3.2. A model of economic growth with local technological spillover effects

This section develops a model of economic growth where local technological spillover effects influence production processes. We consider an economy with a finite set $I = \{1, 2, \ldots, N\}$ of infinitely-lived industries. Each industry $i \in I$ consists of many small, identical firms. Aggregate behavior is thus proportional to the behavior of a representative company. Following Durlauf (1993), we assume that all industries produce an identical output good. Its price is normalized to one. Industries are distinguished by their respective production technologies $\theta_t^i \in [0,1]$. Once a production technology is chosen, labor is the only input. Labor supply is totally inelastic, and $w_t \in [0,1]$ denotes the economy wide wage level in period t. Each industry i chooses a sequence $\{\theta_t^i, l_t^i\}$ of production technologies and labor demands in order to maximize expected profits:

$$\max_{\{\theta_t^i, l_t^i\}} \mathbb{E}\sum_{t=1}^{\infty} \beta^t\left(Y_t^i - w_t l_t^i\right),$$

where Y_t^i denotes the industry's output in period t. Labor can be hired in continuous quantities, $l_t^i \in [0,1]$, and local technological spillovers affect the production processes. The set of companies whose production technologies affect the output of firm i is denoted by $N(i) \subset I$. Specifically, production occurs instantaneously, and the firm produces the output

$$Y_t^i = F\left(l_t^i, \theta_t^i, \left\{\theta_t^j\right\}_{j \in N(i)}\right).$$

The players act non-cooperatively in that they do not take account of their influence on the production of others. No markets exist that allow industries to coordinate; firms cannot be compensated for choosing production technologies that expand the output of the entire economy.

Wages for the period t are fixed at the end of date $t-1$. Wage claims depend on the average labor demand in the preceding period, and on random external conditions like inflation or growth rates. More precisely, we assume that

$$w_t \sim Q\left(w_{t-1}, \frac{1}{N}\sum_{i=1}^{N} l_{t-1}^i; \cdot\right)$$

for some stochastic kernel Q from $[0,1]^2$ to $[0,1]$. Thus, in a large economy the impact of an individual industry on the level of wages is weak. Managers observe w_t before deciding how many workers to hire and which production technology to implement in period t: given w_t, company $i \in I$ takes the action $(l_t^i, \theta_t^i) = \tau^i(w_t)$. Such an assumption is justified if we think of θ^i as a measure for labor intensity. The higher the wages, the more profitable it is to implement a less labor intensive production technology.

The game has a Lipschitz continuous equilibrium if the technological spillover effects are weak enough and if the impact of an individual industry on the wage level is not too strong, i.e., if, for instance, N is large enough. If, in addition, the laws $Q(w, l; \cdot)$ have strictly positive densities with respect to λ on $[0,1]$, then the Markov chain $\{w_t\}$ converges in distribution to a unique limiting measure. However, in the presence of positive technological spillovers, significant multiplier effects may arise both in the unemployment rates and in aggregate output: due to the interactive structure of the economy, the per capita response in labor demand to an economic shock leading to high wages may considerably exceed individual responses in models without local interactions. Thus, even if the overall behavior of process $\{w_t\}$ and hence the overall behavior of labor demand is ergodic, we may still observe large fluctuations in unemployment rates.

3.3. Dynamic production games

Our last application of discounted stochastic games with weakly interacting agents deals with dynamic extensions of the input game discussed in Cooper and John (1988); see also Diamond (1982). There is a set $I = \{1, 2, \ldots, N\}$ of infinitely lived agents sharing a production process. In each period $t \in \mathbb{N}$, the player $i \in I$ bears an effort $x_t^i \in [0,1]$ in the production of a public good at a cost $c(x^i)$. Here $c : [0,1] \to \mathbb{R}$ is a strictly convex cost function. The resulting output is $f(x_t^i, x_t^{-i}, \xi_t)$ where $\xi_t \in [0,2]$ is a parameter that determines the productivity of the agents' choices. The case where ξ_t is an *observable* quantity and where the players take their actions in reaction to ξ_t is analyzed in Section 3.3.1. In such a situation the game's state space is $M = [0,2]$. Thus, under suitable smoothness conditions the existence of equilibria can be established by means of Theorem 2.3. If the productivity parameter is *unobservable*, the analysis becomes more involved. In Section 3.3.2 we consider a game where the agents can only estimate the distribution μ_t of ξ_t before making their choices. In this case the game's state space is the set $\mathcal{P}$ of all probability measures on $[0,2]$ equipped with the total variation norm $\|\cdot\|_V$.

The total variation distance between two probability measures $\nu, \hat{\nu}$ on $[0,2]$ is given by $\|\nu - \hat{\nu}\|_V := \sup_{A\in\mathcal{B}} |\nu(A) - \hat{\nu}(A)|$, where $\mathcal{B}$ denotes the Borel-σ field on $[0,2]$.

3.3.1. Games with observable productivity parameters

Let us first assume that the agents are able to observe the actual productivity parameter. In this case we describe the players' preferences by a utility function of the form

$$U^i(x,\xi) = u\big(f\big(x^i, x^{-i}, \xi\big), c\big(x^i\big)\big).$$

We assume that the conditional distribution of the productivity parameter depends on the average effort:

$$Q(\xi, x; \cdot) = h\left(\frac{\xi + x^1 + \cdots + x^N}{2+N}\right) Q_1(\cdot) + \left[1 - h\left(\frac{\xi + x^1 + \cdots + x^N}{2+N}\right)\right] Q_2(\cdot).$$

Here $h:[0,1] \to [0,1]$ is a twice continuously differentiable function that satisfies $h'' \in [0,1]$, and Q_k has a bounded density q^k with respect to λ on $[0,2]$. We have

$$\big\|q_{i,j}(\xi, x, \cdot)\big\|_{L^1} \leqslant \frac{\|h''\|_\infty}{(2+N)^2} \int \big(q^1(\eta) + q^2(\eta)\big)\mu(\mathrm{d}\eta) \leqslant \frac{2}{(2+N)^2}.$$

Under differentiability conditions on the utility, on the cost and on the production function, it is straightforward to show that the game has an equilibrium if the cost function is sufficiently convex. As an illustration we consider the specific case

$$U^i(\xi, x) = \xi x^i \sum_{j\neq i} x^j - c\big(x^i\big), \quad \text{where} \quad c\big(x^i\big) = 4\big(1 + x^i\big)^3$$

and where the law of motion depends in a linear manner on the agents' efforts:

$$Q(\xi, x; \cdot) = \frac{\xi + x^1 + \cdots + x^N}{2+N} Q_1(\cdot) + \left(1 - \frac{\xi + x^1 + \cdots + x^N}{2+N}\right) Q_2(\cdot).$$

Thus, high efforts and a high productivity parameter make it more likely that the new productivity parameter is chosen according to the probability distribution Q_1. Since

$$q_{i,j}(\xi, x, \eta) = 0, \qquad \frac{\partial^2}{\partial (x^i)^2} U^i(\xi, x) = -24\big(1 + x^i\big) \leqslant -24,$$
$$\frac{\partial^2}{\partial x^i \partial x^j} U^i(\xi, x) = \xi \leqslant 2,$$

the MSI condition holds if $2(N-1) < 24$, i.e., if $N \leqslant 12$. Thus, the game has a Lipschitz continuous equilibrium if at most 12 players participate in the game. Under the additional assumption that Q_1 stochastically dominates Q_2 the game is supermodular. In this case our result can also be derived from Theorem 4.6 in Curtat (1996). Our method allows us to derive existence results without imposing monotonicity conditions on the law of motion.

Glaeser and Scheinkman (2000) discuss the case where an agent's utility depends on the average action taken by all the other agents. In our current setup, this means that

$$U^i(\xi, x) = \xi x^i \frac{\sum_{j\neq i} x^j}{N-1} - 4\big(1 + x^i\big)^3.$$

Let $\beta^i = 0.9$ for all $i \in I$, and consider the more general case where h is not the identity. Since $\|U^i\|_\infty \leqslant 32$ and because $\beta/(1-\beta) = 9$, the weak interaction condition (7) holds if $2 + 9 * 32/(2 + N) < 24$. This inequality is satisfied for all $N \geqslant 12$. Thus, the game has a Lipschitz continuous Nash equilibrium if at least 12 players participate in the game.

3.3.2. Games with unobservable productivity parameters

Let us now consider the case where the actual productivity parameter ξ_t is unobservable. The players only know its distribution μ_t. Preferences are described by utility functions $U^i : \underline{\overline{X}} \times \mathcal{P} \to \mathbb{R}$ of the form

$$U^i(x_t, \mu_t) = x_t^i [\mathbb{E}_{\mu_t}\xi - \sigma \mathbb{V}_{\mu_t}\xi] \sum_{j \in I} x_t^j - 4\big(1 + x^i\big)^3$$

where $\mathbb{E}_{\mu_t}\xi$ and $\mathbb{V}_{\mu_t}\xi$ denote the mean and the variance of the random variable ξ under the law μ_t, respectively. The parameter σ specifies the agents' common degree of risk aversion. Thus, in a static model the players would be mean-variance maximizers.

We assume that the agents can control the dynamics of the sequence of distributions $\{\mu_t\}$. More precisely, we fix stochastic kernels Q_1 and Q_2 on $\mathcal{P}$. Given a probability measure $\mu \in \mathcal{P}$, the law $Q_1(\mu; \cdot)$ is concentrated on a set of probability measures under which ξ has a high mean, but also a high variance. The law $Q_2(\mu; \cdot)$ is concentrated on a set of measures under which the productivity parameter has both a lower mean and a lower variance. Specifically,

$$Q(\mu_t, x_t; \cdot) = \frac{x_t^1 + \cdots + x_t^N}{N} Q_1(\mu_t; \cdot) + \left(1 - \frac{x_t^1 + \cdots + x_t^N}{N}\right) Q_2(\mu_t; \cdot).$$

Thus, a high effort increases the expected productivity, but also its variance. If the agents do not observe the actual productivity parameter, but only its distribution, the game's state is $\mathcal{P}$ which is not a Euclidean space. In order to derive sufficient conditions for the existence of Lipschitz continuous Nash equilibria, we apply (12). Due to the linear structure of the transition kernel Q we may choose $\widehat{L}^{i,j}(\mu, \eta) = 0$ for all $i, j \in I$. The utility functions are α-concave with $\alpha = 24$ and

$$L^{i,j}(\mu) = |\mathbb{E}_\mu \xi - \sigma \mathbb{V}_\mu \xi|.$$

Hence the weak interaction condition (12) holds if $(N-1)\sup_{\mu \in \mathcal{P}} |\mathbb{E}_\mu \xi - \sigma \mathbb{V}_\mu \xi| < 24$. Typically, $|\mathbb{E}_\mu \xi - \sigma \mathbb{V}_\mu \xi| < 2$. Therefore, the game may have an equilibrium for $N > 12$. The additional uncertainty about the true productivity parameter reduces the impact of an individual agent on the utility of others. Hence we can possibly allow for more players to participate in the game and still guarantee the existence of equilibria.

4. Lipschitz continuous equilibria in a static one-shot game

This section prepares the proofs of our main results by proving the existence of Lipschitz continuous Nash equilibria in a certain class of one-shot games.

Since the agents' instantaneous utility functions are bounded, we may with no loss of generality assume that $U^i \geqslant 0$ for all $i \in I$. We introduce the vector $u = (u^i)_{i \in I}$ with components $u_i := \|U^i\|_\infty$ and denote by $(B_u(M, \mathbb{R}^N), \|\cdot\|_\infty)$ the Banach space of all non-negative measurable functions $f : M \to \mathbb{R}^N_+$ satisfying $\|f^i\|_\infty \leqslant u^i$. To each such *average continuation function* we associate the *reduced one-shot game* $\Sigma_f := (I, (\underline{\overline{X}}^i, U^{i,f}), \xi)$ with payoff functions

$$U^{i,f}(\xi, x) = \left(1 - \beta^i\right)U^i(\xi, x) + \beta^i \int_M f^i(\eta)q(\xi, x, \eta)\mu(\mathrm{d}\eta). \tag{16}$$

The following lemma shows that the reduced game Σ_f has a unique Nash equilibrium $g_f(\xi)$, due to the weak interaction condition. Moreover, the equilibrium map $g_f : M \to \underline{\overline{X}}$ is Lipschitz continuous with a constant that can be chosen independently of the specific average continuation function. This property turns out to be the key to the proof of Theorem 2.10.

Lemma 4.1. *Under the Assumptions of Theorem* 2.10 *the following holds for every* $f \in B_u(M, \mathbb{R}^N)$:

(i) *For each* $\xi \in M$ *and* $x^{-i} \in \underline{\overline{X}}^{-i}$, *the map* $x^i \mapsto U^{i,f}(\xi, x^i, x^{-i})$ *is*

$$\hat{\alpha}^i(\xi) = \left(1 - \beta^i\right)\alpha^i(\xi) - u^i\beta^i\widehat{L}^{i,i}(\xi)$$

concave on $\underline{\overline{X}}^i$, *and* $\inf_\xi \hat{\alpha}^i(\xi) > 0$.

(ii) *The conditional best reply* $g^i_f(\xi, x^{-i})$ *of player* $i \in I$ *depends in a Lipschitz continuous manner on the actions of his competitors. More precisely, we have*

$$\left\|g^i_f\left(\xi, x^{-i}\right) - g^i_f\left(\xi, y^{-i}\right)\right\|_i \leqslant \frac{(1 - \beta^i)L^{i,j}(\xi) + u^i\beta^i\widehat{L}^{i,j}(\xi)}{\hat{\alpha}^i(\xi)}\left\|x^j - y^j\right\|_j \tag{17}$$

if $x^k = y^k$ *for all* $k \neq j$. *Moreover, there exists* $\widetilde{L} < \infty$ *such that*

$$\left\|g^i_f\left(\xi_1, x^{-i}\right) - g^i_f\left(\xi_2, x^{-i}\right)\right\|_i \leqslant \widetilde{L}\|\xi_1 - \xi_2\|_M \tag{18}$$

for all $\xi_1, \xi_2 \in M$ *and each* $x^{-i} \in \underline{\overline{X}}^{-i}$.

(iii) *The reduced game* Σ_f *has a unique equilibrium* $g_f(\xi) = \{g^i_f(\xi)\}_{i\in I} \in \underline{\overline{X}}$.

(iv) *The mapping* $\xi \mapsto g^i_f(\xi)$ *is Lipschitz continuous uniformly in* $f \in B_u(M, \mathbb{R}^N)$. *That is, there exists* $L_g < \infty$ *such that*

$$\left\|g^i_f(\xi_1) - g^i_f(\xi_2)\right\|_M \leqslant L_g\|\xi_1 - \xi_2\|_M$$

for all average continuation functions $f \in B_u(M, \mathbb{R}^N)$.

(v) *The map* $f \mapsto g^i_f(\cdot)$ *from* $B_u(M, \mathbb{R}^N_+)$ *to* $B(M, \underline{\overline{X}})$ *is continuous.*

Proof. (i) Let us fix an average continuation function f, an action profile $x^{-i} \in \underline{\overline{X}}^{-i}$ and a state $\xi \in M$. Because U^i is Lipschitz continuous and because of (A.1), it is enough to show that

$$U^{i,f}_1\left(\xi, x^i, x^{-i}; x^i - \hat{x}^i\right) - U^{i,f}_1\left(\xi, \hat{x}^i, x^{-i}; x^i - \hat{x}^i\right) \leqslant -\hat{\alpha}^i(\xi)\left\|x^i - \hat{x}^i\right\|^2_i \tag{19}$$

for all $x^i, \hat{x}^i \in \underline{X}^i$. In order to prove (19), we put

$$F^i(\xi, x^i, x^{-i}) := \int_M f^i(\eta) q(\xi, x^i, x^{-i}, \eta)\mu(\mathrm{d}\eta).$$

By Assumption 2.8(iii), the directional derivative $F_1^i(\xi, x^i, x^{-i}; x^i - \hat{x}^i)$ of the map $x^i \mapsto F^i(\xi, x^i, x^{-i})$ at (ξ, x) in the direction $x^i - \hat{x}^i$ exists and satisfies

$$\left|F_1^i(\xi, x^i, x^{-i}; x^i - \hat{x}^i) - F_1^i(\xi, \hat{x}^i, x^{-i}; x^i - \hat{x}^i)\right| \leqslant u^i \widehat{L}^{i,i}(\xi) \|x^i - \hat{x}^i\|_i^2.$$

Since $U^i(\xi, \cdot, x^{-i})$ is $\alpha^i(\xi)$-concave on $\underline{X}^i$ we have

$$U_1^i(\xi, x^i, x^{-i}; x^i - \hat{x}^i) - U_1^i(\xi, \hat{x}^i; x^{-i}; x^i - \hat{x}^i) \leqslant -\alpha^i(\xi) \|x^i - \hat{x}^i\|_i^2.$$

Thus, the concavity condition (19) is satisfied if $(1 - \beta^i)\alpha^i(\xi) > \beta^i u^i \widehat{L}^{i,i}(\xi)$. This, however, as well as $\inf_{\xi \in M} \hat{\alpha}^i(\xi) > 0$ follows from the MSI condition.

(ii) Since an agent's utility function is strongly concave with respect to his own action, his conditional best reply given the choices of his competitors is uniquely determined. To establish the quantitative bound (17) on the dependence of player i's best reply on the behavior of all the other agents, we fix a player $j \neq i$ and action profiles x^{-i} and y^{-i} which differ only at the jth coordinate. Under the assumptions of Theorem 2.10 the directional derivative $U_1^{i,f}(\xi, x^i, x^{-i}; h^i)$ of the map $x^i \mapsto U^{i,f}(\xi, x^i, x^{-i})$ at (ξ, x) in the direction $h^i \in H^i$ satisfies

$$\begin{aligned}
&\left|U_1^{i,f}(\xi, x^i, x^{-i}; h^i) - U_1^{i,f}(\xi, x^i, y^{-i}; h^i)\right| \\
&\quad \leqslant \left\{(1 - \beta^i)L^{i,j}(\xi) + \beta^i u^i \widehat{L}^{i,j}(\xi)\right\} \|x^j - y^j\|_j \|h^i\|_i.
\end{aligned}$$

Thus, (17) follows from Theorem A.1. Our estimate (18) follows from similar considerations.

(iii) The existence of an equilibrium in pure strategies for the static game Σ_f follows from strict concavity of the utility functions $U^{i,f}$ with respect to the player's own actions along with compactness of the action spaces using standard fixed point arguments. Uniqueness can be seen as follows: in view of the MSI condition,

$$\widehat{L} := \sup_{i,\xi} \sum_{j \neq i} \frac{(1 - \beta^i)L^{i,j}(\xi) + \beta^i u^i \widehat{L}^{i,j}(\xi)}{\hat{\alpha}^i(\xi)} < 1.$$

Thus, given the action profiles x^{-i} and y^{-i} of player i's competitors, (17) yields

$$\|g_f^i(\xi, x^{-i}) - g_f^i(\xi, y^{-i})\|_i \leqslant \widehat{L} \max_j \|x^j - y^j\|_j.$$

For $x \neq y$, we therefore obtain

$$\max_i \|g_f^i(\xi, x^{-i}) - g_f^i(\xi, y^{-i})\|_i < \max_i \|x^i - y^i\|_i.$$

Thus, the map $x \mapsto (g_f^i(\xi, x^{-i}))_1^N$ has at most one fixed point. This proves uniqueness of equilibria in Σ_f.

(iv) Let $g^f(\xi)$ be an equilibrium. Then $g^i_f(\xi) = g^i_f(\xi, \{g^j_f(\xi)\}_{j\neq i})$, and so

$$\begin{aligned}\|g^i_f(\xi_1) - g^i_f(\xi_2)\|_i &\leqslant \|g^i_f(\xi_1, \{g^j_f(\xi_1)\}_{j\neq i}) - g^i_f(\xi_1, \{g^j_f(\xi_2)\}_{j\neq i})\|_i \\ &\quad + \|g^i_f(\xi_1, \{g^j_f(\xi_2)\}_{j\neq i}) - g^i_f(\xi_2, \{g^j_f(\xi_2)\}_{j\neq i})\|_i \\ &\leqslant \widehat{L} \max_j \|g^j_f(\xi_1) - g^j_f(\xi_2)\|_j + \widetilde{L}\|\xi_1 - \xi_2\|_M.\end{aligned}$$

This yields

$$\|g^i_f(\xi_1) - g^i_f(\xi_2)\|_i \leqslant \frac{\widetilde{L}}{1-\widehat{L}}\|\xi_1 - \xi_2\|_M,$$

and so the equilibrium mapping $g_f : M \to \underline{\overline{X}}$ is Lipschitz continuous which a constant that does not depend on the average continuation function f.

(v) In order to prove the last assertion we fix $\xi \in M$ and $x^{-i} \in \underline{\overline{X}}^{-i}$ and apply Theorem A.1 to the map

$$(x^i, f) \mapsto U^{i,f}(\xi, x^i, x^{-i}).$$

Due to Assumption 2.8(i) we have for all $f, g \in B_u(M, \mathbb{R}^N)$ that

$$|U_1^{i,f_1}(\xi, x^i, x^{-i}; h^i) - U_1^{i,f_2}(\xi, x^i, x^{-i}; h^i)| \leqslant \beta^i \|f_1 - f_2\|_\infty \|h^i\|_i,$$

and so Theorem A.1 shows that

$$\|g^i_{f_1}(\xi, x^{-i}) - g^i_{f_2}(\xi, x^{-i})\|_i \leqslant \frac{\beta^i}{\inf_\xi \hat{\alpha}^i(\xi)}\|f_1 - f_2\|_\infty.$$

Thus, similar arguments as in the proof of (iii) yield the assertion. □

The previous lemma allows us to discuss the connection between our *Moderate Social Influence* assumption and the monotonicity conditions in Curtat (1996) in greater detail. Basically, Curtat (1996) assumes that $\underline{\overline{X}}^i$ and M are compact intervals, and that the transition law $Q(\xi, x; \cdot)$ has "doubly stochastically increasing differences in x and ξ." Inter alias this means that, for any increasing function $f : M \to \mathbb{R}$, the map

$$x \mapsto \int f(\eta) q(\xi, x, \eta) \nu(\mathrm{d}\eta) \tag{20}$$

has doubly increasing differences in x and ξ. Thus, there is a Lipschitz continuous function $\phi : \mathbb{R} \to \mathbb{R}$ such that the map

$$x \mapsto \int f(\eta) q(\xi, \phi(\xi)\mathbf{1} - x;\ \mathrm{d}\eta) \nu(\mathrm{d}\eta) \tag{21}$$

has increasing differences in x and ξ. Here $\mathbf{1}$ denotes the vector $(1, 1, \ldots, 1)$ in $\mathbb{R}^N$. For the proof of Theorem 4.2 in Curtat (1996) it is now essential that the Lipschitz continuous "change of variables" ϕ can be chosen independently of f. In his Theorem 2.4, this author essentially shows that a sufficiently smooth function $F : \underline{\overline{X}} \times \mathbb{R} \to \mathbb{R}$ has doubly increasing differences, if and only if it has increasing differences in x and ξ and if it satisfies the diagonal dominance condition

$$f_x^i := \frac{\partial^2 F}{\partial^2 x^i} + \sum_{j \neq i} \frac{\partial^2 F}{\partial x^i \partial x^j} \leqslant 0.$$

Applied to the map defined by (20), such a diagonal dominance condition holds uniformly in f, if the densities depend linearly on the players' actions as in (8) above. However, we are unaware of any general method that allows us to verify Curtat's condition in more general settings without explicitly specifying the law of motion. This motivated our MSI condition.

5. Proofs of the main results

This section proves Theorem 2.10 and Corollary 2.13. In a first step we establish the existence of a Lipschitz continuous Nash equilibrium for Σ under the additional assumption that $M \subset H$ is compact. For the average continuation function $f \in B_u(M, \mathbb{R}^N)$, we denote by $g_f(\xi)$ the unique equilibrium in the one-shot game Σ_f, and introduce an operator T on $B_u(M, \mathbb{R}^N)$ by

$$(Tf)^i(\xi) = (1 - \beta^i) U^i(\xi, g_f(\xi)) + \beta^i \int_M f^i(\eta) q(\xi, g_f(\xi), \eta) \mu(\mathrm{d}\eta). \tag{22}$$

Assume that T has a fixed point, F. A standard argument in discounted dynamic programming shows that the action profile $g_F(\xi)$ is an equilibrium in the non-zero sum stochastic game Σ. The equilibrium payoff to player $i \in I$ is given by $F^i(\xi)/(1 - \beta^i)$, and the map $g_F : M \to \overline{X}$ is Lipschitz continuous, due to Lemma 4.1.

In order to prove Theorem 2.10 it is therefore enough to establish the existence of a fixed point of the operator T. To this end, we will need the following basic properties of T.

Lemma 5.1. *Under the assumptions of Theorem* 2.10 *the following holds*:

(i) *For all $f \in B_u(M, \mathbb{R}^N)$, the mapping $\xi \mapsto (Tf)(\xi)$ is Lipschitz continuous.*
(ii) *The operator T is continuous in the sense that* $\lim_{n\to\infty} \|Tf - Tf_n\|_\infty = 0$ *whenever* $\lim_{n\to\infty} \|f - f_n\|_\infty = 0$.

Proof. (i) It follows from Lipschitz continuity of the utility functions and the densities that

$$\begin{aligned}\left|(Tf)^i(\xi_1) - (Tf)^i(\xi_2)\right| \leqslant{}& \left[(1 - \beta^i)L + \beta^i L u^i\right] \\ & \times \left(\|\xi_1 - \xi_2\|_M + \left\|g_f(\xi_1) - g_f(\xi_2)\right\|_M\right).\end{aligned} \tag{23}$$

Thus, Lipschitz continuity of the mapping $\xi \mapsto g_f(\xi)$ yields Lipschitz continuity of $(Tf)^i$.

(ii) In order to prove continuity of T in the topology of uniform convergence, we fix functions $f_n \in B_u(M, \mathbb{R}^N)$ that converge uniformly to f. Lemma 4.1(v) yields $\lim_{n\to\infty} \|g_{f_n} - g_f\|_\infty = 0$. Thus, Lipschitz continuity of the reward functions and the densities gives us

$$\big|(Tf_n)^i(\xi) - (Tf)^i(\xi)\big| \leqslant \big(1-\beta^i\big)L\|g_{f_n} - g_f\|_\infty + \beta^i\big\{\big\|f_n^i - f^i\big\|_\infty + u^i L\|g_{f_n} - g_f\|_\infty\big\},$$

and so

$$\lim_{n\to\infty} \|Tf_n - Tf\|_\infty = 0.$$

This finishes the proof. □

Let L_g be the common Lipschitz constant of the maps $g_f : M \to \underline{X}$ and define

$$L^* := \max\big\{\big[\big(1-\beta^i\big)L + \beta^i L u^i\big](1+L_g) \colon\ i \in I\big\}.$$

We introduce the class $\mathcal{L}(L^*, u)$ of all functions $f \in B_u(M, \mathbb{R}^N)$ which are Lipschitz continuous with constant L^*. For $f \in \mathcal{L}(L^*, u)$ we obtain from Lemma 5.1(i) that

$$\big|Tf_i(\xi_1) - Tf_i(\xi_2)\big| \leqslant L^* \|\xi_1 - \xi_2\|_M.$$

Thus, T maps the set $\mathcal{L}(L^*, u)$ continuously into itself. We are now ready to prove the main result of this section.

Proof of Theorem 2.10. Due to the theorem of Arzela and Ascoli, the convex set $\mathcal{L}(L^*, u)$ is compact with respect to the topology of uniform convergence. Since the operator T maps to set $\mathcal{L}(L^*, u)$ continuously into itself, it has a fixed point F^* by Schauder's theorem, and g_{F^*} is a Lipschitz continuous equilibrium of the non-cooperative stochastic game Σ. □

Before we prove the existence result for Lipschitz continuous equilibria in non-cooperative stochastic games with unbounded state spaces, we recall the following:

Lemma 5.2. *Let $\{F_n\}$ be a sequence of real-valued continuous functions on M that converges to $F : M \to \mathbb{R}$ uniformly on bounded sets. Let $\{\mu_n\}$ be a sequence of probability measures that converges weakly to μ. If $\sup_n \|F_n\|_\infty < \infty$, then*

$$\lim_{n\to\infty} \int F_n \,\mathrm{d}\mu_n = \int F \,\mathrm{d}\mu.$$

Proof. Since $\mu_n \to \mu$ weakly, we have $\int F \,\mathrm{d}\mu_n \to \int F \,\mathrm{d}\mu$ as $n \to \infty$. Moreover, by Prohorov's theorem (Breiman, 1968) there exists, for each $\epsilon > 0$, a compact set K such that $\mu_n(K) \geqslant 1 - \epsilon$. Thus, for all sufficiently large $n \in \mathbb{N}$ we obtain

$$\begin{aligned}
&\left|\int F_n \,\mathrm{d}\mu_n - \int F \,\mathrm{d}\mu\right| \\
&\quad \leqslant \left|\int (F - F_n)\,\mathrm{d}\mu_n\right| + \left|\int F(\mathrm{d}\mu_n - \mathrm{d}\mu)\right| \\
&\quad \leqslant 2\sup_n \|F_n\|_\infty \mu_n\big(K^c\big) + \left|\int_K (F - F_n)\,\mathrm{d}\mu_n\right| + \left|\int F(\mathrm{d}\mu_n - \mathrm{d}\mu)\right|
\end{aligned}$$

$$\leqslant 2 \sup_n \|F_n\|_\infty \epsilon + \sup_{x \in K} \big|F_n(x) - F(x)\big| + \epsilon$$
$$\leqslant 2\epsilon\big(\sup_n \|F_n\|_\infty + 1\big).$$

This proves the assertion because $\sup_n \|F_n\|_\infty < \infty$ and because $\epsilon > 0$ is arbitrary. □

We are now ready to prove Corollary 2.13.

Proof of Corollary 2.13. Let $T_n : B_u(M_n, \mathbb{R}^N_+) \to B_u(M_n, \mathbb{R}^N_+)$ be defined by

$$(T_n f_n)^i(\xi) = \big(1 - \beta^i\big) U^i\big(\xi, g_{f_n}(\xi)\big) + \beta^i \int_{M_n} f_n^i(\eta) q_n\big(\xi, g_{f_n}(\xi), \eta\big) \mu(\mathrm{d}\eta).$$

Here, $g_{f_n}(\xi)$ denotes the unique equilibrium in the one-shot game Σ_{f_n} with average continuation function $f_n \in B_u(M_n, \mathbb{R}^N)$ and densities q_n. Let F_n be a fixed point of T_n. Due to our Lemmas 4.1 and 5.1, the mappings $g_{F_n} : M_n \to \underline{X}$ and $F_n : M_n \to \mathbb{R}^N$ $(n \in \mathbb{N})$ are Lipschitz continuous with common Lipschitz constants. In particular, the sequence $\{(g_{F_n}, F_n)\}$ is equicontinuous, and so the theorem by Arzela and Ascoli yields a subsequence (n_k) and Lipschitz continuous functions $F : M \to \mathbb{R}$ and $g : M \to \overline{\underline{X}}$ such that

$$\lim_{k \to \infty} \big|F_{n_k}(\xi) - F(\xi)\big| = 0 \quad \text{and} \quad \lim_{k \to \infty} \big|g_{F_{n_k}}(\xi) - g(\xi)\big| = 0$$

uniformly on compact sets.

Since the utility functions are uniformly bounded, weak convergence of the sequence of probability measures $\{\mu_{n_k}(\xi, x; \cdot)\}$ defined in (14) to $\mu(\xi, x; \cdot)$ yields

$$\lim_{k \to \infty} \int_H F_{n_k}^i(\eta) q_{n_k}\big(\xi, g_{F_{n_k}}(\xi), \eta\big) \mu(\mathrm{d}\eta) = \int_H F^i(\eta) q\big(\xi, g(\xi), \eta\big) \mu(\mathrm{d}\eta),$$

due to Lemma 5.2. We deduce that

$$F^i(\xi) = \big(1 - \beta^i\big) U^i\big(\xi, g(\xi)\big) + \beta^i \int_H F^i(\eta) q\big(\xi, g(\xi), \eta\big) \mu(\mathrm{d}\eta).$$

It is easily seen that $g(\xi)$ is an equilibrium in the one-shot game Σ_F with densities q. Thus, g is a Lipschitz continuous Nash equilibrium of the stochastic game Σ with unbounded state space. □

6. Conclusion

We established existence of Lipschitz continuous equilibria in stationary strategies for a class of stochastic games with weakly interacting players. Unlike the method in Curtat (1996), our proof did not need Topkis' (1978) monotonicity theorem. This allowed us to go beyond the class of supermodular games analyzed in Amir (1996) and Curtat (1996). Instead, our approach was based on an extension of the *Moderate Social Influence* condition in Glaeser and Scheinkman (2000) to dynamic games. We reduced the dynamic decision problem to a static game through the introduction of average continuation functions in order to view an agent's decision problem as a parameter dependent optimization problem. Using a result by Montrucchio (1987), we proved that the optimization problems have optimal solutions that are Lipschitz continuous functions of the parameters. Combining these results with our weak interaction condition, we showed that the reduced one shot game has a unique equilibrium that is Lipschitz continuous in the state variable. Since the Lipschitz constant could be chosen independently of the specific average continuation function, the existence of Lipschitz continuous equilibria could be established using standard results from the theory of dynamic programming. For the case of compact state spaces we also proved existence of ergodic equilibria. Our results provide a general framework for analyzing dynamic models of non-market interactions.

Several avenues are open for future research. Firstly, our goal was to provide a general and flexible mathematical framework within which existence of continuous equilibria can be shown. But it is clearly desirable to weaken our *Moderate Social Influence* condition by analyzing special classes of stochastic games where the set of average continuation functions can a priori be restricted to a proper subset of $B_u(M, \mathbb{R}^N)$. In such a situation, much weaker conditions may actually apply. Secondly, there is no reason to expect uniqueness of equilibria. For the dynamic growth model studied in Section 3.2 this means that the economy may well get stuck in an inefficient equilibrium. In general it would be interesting to study welfare properties of different equilibria in the context of specific models. Thirdly, the class of local interaction games analyzed in Section 3.1 should be generalized to games where an agent's utility does not only depend on his current action, but also on past choices as in Bisin et al. (2002). Such a situation cannot be analyzed by our method.

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Appendix A. α-concavity and parameterized optimization problems

In this appendix we recall the notion of α-concave functions and a characterization of α-concavity in terms of partial derivatives. We also recall a result on Lipschitz continuous dependence of solutions on parameterized optimization problems, due to Montrucchio (1987). Throughout, Y denotes a convex subset of a Hilbert space H, and $\alpha > 0$.

A function $f : Y \to \mathbb{R}$ is called *α-concave* if the map $y \mapsto f(y) + \frac{1}{2}\alpha\|y\|^2$ is concave on Y. In the differentiable case, there are simple criteria to verify α-concavity. For example, if f is concave and twice differentiable on an open set Y_1 containing Y, then f is α-concave whenever

$$\left|y^t D^2 f(y_1) y\right| \geqslant \alpha \|y\|^2 \quad \text{for all } y_1 \in Y_1 \text{ and } y \in Y.$$

A twice differentiable function $f:[a,b] \to \mathbb{R}$ is α-concave if $f'' \leqslant -\alpha$. More generally, α-concavity can be characterized in terms of directional derivatives.

To this end, recall that a finite function $f: Y \to \mathbb{R}$ is called differentiable at $y \in Y$ in the feasible direction $h \in H$ if $y + th \in Y$ for some $t > 0$ and if the limit

$$f'(y; h) := \lim_{t \downarrow 0} \frac{1}{t}\bigl(f(y+th) - f(y)\bigr)$$

exists and is finite. The map f is called differentiable if it is differentiable at all $y \in Y$ in all feasible directions $h \in H$. By Propositions 4.8 and 4.12 in Vival (1983), a finite and differentiable function f is α-concave if and only if

$$f \text{ is Lipschitz continuous} \quad \text{and}$$
$$f'(y_1; y_1 - y_2) - f'(y_2; y_1 - y_2) \leqslant -\alpha \|y_1 - y_2\|^2. \tag{A.1}$$

The proof of our main theorem uses the following results which appears as Theorem 3.1 in Montrucchio (1987).

Theorem A.1. *Let X be a closed and convex subset of some Hilbert space $(H_1, \|\cdot\|_1)$ and let Y be a convex subset of a normed space $(H_2, \|\cdot\|_2)$. Let $F: X \times Y \to \mathbb{R}$ be a finite function which satisfies the following conditions*:[3]

(i) *For all $y \in Y$, the map $x \mapsto F(x,y)$ is α-concave and upper-semicontinuous on X.*
(ii) *For all feasible $h \in H$, the directional derivative $F_1(x,y;h)$ of F at (x,y) in the direction h satisfies the Lipschitz continuity condition*

$$\bigl|F_1(x, y_1; h) - F_1(x, y_2; h)\bigr| \leqslant L\|y_1 - y_2\|_2 \|h\|_1$$

for all $y_1, y_2 \in Y$ and all $x \in X$.

Under the above assumptions there exists a unique map $\theta: Y \to X$ that satisfies $\sup_{x \in X} = F(x,y) = F(\theta(y), y)$. *Moreover, θ is Lipschitz continuous and*

$$\bigl|\theta(y_1) - \theta(y_2)\bigr| \leqslant \frac{L}{\alpha}\|y_1 - y_2\|_2$$

for all $y_1, y_2 \in Y$.

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[3] Montrucchio (1987) formulated this theorem under the additional assumption of Y being a closed and convex subset of a Hilbert space H_2. His proof, however, shows that this assumption is redundant.

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