

# A Maximum Principle approach to a deterministic Mean Field Game of Control with Absorption \*

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## Abstract

We study a deterministic mean field game on finite and infinite time horizons arising in models of optimal exploitation of exhaustible resources. The main characteristic of our game is an absorption constraint on the players' state process. As a result of the state constraint the optimal time of absorption becomes part of the equilibrium. Using Pontryagin's maximum principle, we prove the existence and uniqueness of equilibria and solve the infinite horizon models in closed form. As players may drop out of the game over time, equilibrium production rates need not be monotone nor smooth.

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## 1 Introduction

This paper establishes existence and uniqueness of equilibrium results for a deterministic mean field game (MFG) arising in models of optimal exploitation of exhaustible resources. MFGs provide a convenient tool for analyzing complex strategic interactions among many players when each individual player has only a small impact on the behavior of other players. In a standard mean field game each player solves a control problem in which an individual player's payoff functional and the dynamics of the controlled state process depend on the empirical distribution of the other players' actions or states. The existence of Nash equilibria in MFGs can be established by solving either a coupled system of two partial differential equations (PDEs), a backward Hamilton-Jacobi-Bellman equation determining the players' utility, and a forward Kolmogorov equation determining the evolution of the distribution of states or actions, or by solving a system of McKean-Vlasov forward-

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backward SDEs where the forward component describes the state dynamics and the backward component describes the dynamics of the adjoint variable. We refer to the monograph [1] for further background.

In the economics literature MFGs are often called anonymous games. First introduced by Rosenthal [29] and Jovanovic and Rosenthal [26], anonymous games have received renewed attention among economists in the last two decades; see [2, 12, 23] and references therein. Huang, Malhamé and Caines [24] and Lasry and Lions [27] independently introduced MFGs into the engineering and mathematical literature. Ever since, MFGs have become an important driver of mathematical innovation, especially in the areas of PDEs and backward stochastic equations. Complementing the theoretical work on MFGs, there is a by now substantial literature where anonymous and mean field games have been successfully applied to an array economic and engineering problems, ranging from network security and traffic networks [18, 30] and systemic risk management [8], to portfolio liquidation [13, 14, 16, 25] and oil and energy production in competitive markets [9, 10, 19, 20, 21].

Oligopoly models of markets with a small number of competitive players that compete on the amount of output they produce go back to the classical work of Cournot [11] in 1838. These have typically been static (or one-period) models, where the existence and construction of a Nash equilibrium have been extensively studied. Dynamic Cournot models for energy production in a competitive market have been proposed and analyzed by many authors in recent years; we refer to [28] for a survey on energy production models. In these models market participants are often endowed with limited amounts of exhaustible resources such as oil, coal or natural gas that they choose to extract for sale. The models may lead to continuous time single player control problems [4], finite player nonzero-sum differential games [22], or continuum mean field games [9, 10]. In the context of nonzero-sum dynamic games between finitely many players the computation of a solution is a challenging problem, typically involving coupled systems of nonlinear PDEs, with one value function per player, and existence theory is sparse. MFGs allow one to handle certain types of competition in the continuum limit of an infinity of small players by solving either systems of PDEs or forward-backward SDEs.

One of the key characteristics of games with exhaustible resources is that resources may run out and change the structure of the game. In stochastic games absorption constraints are challenging to incorporate and the literature on stochastic games with absorption is sparse.<sup>1</sup> Specific stochastic MFGs with absorption have recently been considered by Campi and coauthors in [5, 6, 7]. Graber and Bensoussan [19] consider a modification of the model of Chan and Sircar [9] with absorption but restrict their state dynamics to a bounded domain. While the restriction to bounded domains simplifies the mathematical analysis it seems undesirable from an economic perspective. The situation is much simpler for deterministic MFGs. As shown in the recent work by Bonnans et al [3] the existence MFG equilibria can be established under rather general mixed state-control and terminal state constraints.

We consider a *deterministic* MFG of optimal exploitation in which players with zero resources drop

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<sup>1</sup>In models of optimal exploitation absorption constraints are often ignored, assuming that resource levels fluctuate randomly and recover as soon as they reach positive levels again. Likewise, in models of optimal portfolio liquidation non-negativity constraints on the stock holding are usually ignored as well.

out of the game. Both finite and the infinite horizon models are studied. As we will see there can be very different equilibrium strategies depending on whether a terminal time is imposed or not. In our model only positive controls are allowed. As a result, we can rephrase the single/representative player model underlying our MFG as a standard convex control problem with state constraints. Due to the state constraint the terminal value of the adjoint process is unknown and part of the solution.<sup>2</sup> It thus seems natural to consider the Hamiltonian of the representative player model only up a candidate optimal exploitation time in terms of which a candidate terminal condition for the adjoint process can be obtained. A straightforward verification argument shows that the candidate exhaustion time and hence the candidate terminal condition are indeed optimal.

Our focus in solving the MFG is hence on the equilibrium exhaustion time. The key observation is that the representative player’s best response function is independent of the mean field equilibrium. More precisely, we prove that the dynamics of the amount of resource extracted by an individual player up to any *given* time by using a strategy that would be optimal if the optimal exhaustion time was equal to that given time is independent of the *equilibrium* mean production rate. Moreover, we show that the dynamics follows an ODE that depends only the initial distribution of the resource levels and the risk free interest rate. The ODE can be solved in closed form. The solution is strictly monotone and the optimal exhaustion time is given by the inverse function. In the infinite horizon case where the equilibrium mean production rate converges to zero as time increases to infinity, this allows us to fully describe the unique MFG equilibrium in terms of the said ODE. In the finite horizon case the equilibrium mean production rate at the terminal time is unknown. This results in an additional fixed point problem that can easily be solved. Thus we provide among the very few explicit solutions to MFGs outside the linear-quadratic framework.

The approach taken in this paper is very different from - and complements - recent work of Bonnans et al [3]. They establish an abstract existence of solutions result for a general class of finite-time MFGs of controls with mixed state-control and terminal state constraints. Short of some technical assumptions such as their boundedness assumption on the initial states, their model contains ours as a special case. Their analysis is based on a sophisticated, yet abstract fixed point argument which makes it difficult to solve MFGs in closed form.<sup>3</sup> By contrast, our analysis is based on an explicit representation of the equilibrium absorption time. This approach is new and does not require sophisticated mathematical methods. Another notable difference is that we first analyze the infinite horizon model and then use that solution to solve the finite horizon case. In [3] only the finite horizon case is considered; using their approach we expect the solution to our infinite horizon model to be obtained by first introducing a discount factor into their model and then taking the limit as the time horizon tends to infinity.

To illustrate our main ideas we first consider in Section 2 the benchmark case of a single monopolist oil producer. We fully characterize optimal exploitation strategies. In particular, we prove that

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<sup>2</sup>This is again very similar to portfolio liquidation models where the terminal condition of the adjoint process is unknown, due to the liquidation constraint; see [14, 15] for details.

<sup>3</sup>As it is often (though not always; see, e.g. the recent work of Fu et al. [17]) the case with MFGs of control Bonnans et al. introduce an auxiliary mapping that allows them “to write the equilibrium problem in a reduced form which is then tractable with a fixed point argument. After reformulation, the equilibrium problem is posed on the set of Borel probability measures on the space of state-costate trajectories”.

full exploitation may not be optimal in finite horizon problems. The MFG of optimal extraction is analyzed in Section 3. We first determine the equilibrium time of exploitation as a function of the competitors' strategies and own initial resources. Subsequently, we determine the equilibrium exploitation times and strategies. We illustrate by two simple examples that equilibrium exploitation rates do need not be monotone, nor smooth.

## 2 The Monopoly Case

As a motivation for our general analysis we illustrate in this section our main ideas in the framework of monopolist oil producer. We fix a time horizon  $T \in (0, \infty)$  and denote the monopolist's resource at time 0 by  $x_0$ . The monopolist extracts the resource according to a measurable rate function (control)  $q : [0, T] \rightarrow [0, \infty)$ . A control is called *admissible* if the corresponding state process

$$X_t^q := x_0 - \int_0^t q_u du$$

is non-negative on  $[0, T]$ . Since only non-negative controls are admissible, the non-negativity of the state process is equivalent to the terminal state constraint  $X_T^q \geq 0$ .

Following [9, 10], we assume that the price function  $p : [0, T] \rightarrow (-\infty, 1]$  when the production rate  $q$  is employed is given by  $p = 1 - q$ . For a given constant discount rate  $r > 0$  the monopolist's discounted revenue is then given by

$$J(q) := \int_0^T e^{-rt} q_t (1 - q_t) dt$$

and the value function is given by

$$u(x_0) := \sup_{q \geq 0, X_T^q \geq 0} J(q). \quad (2.1)$$

The Hamiltonian associated with our control problem is given by

$$H(x, q, y) = q(1 - q) - yq$$

where  $y$  is the *adjoint variable*. The end-point Lagrangian is given by

$$L(y_T, \psi) = \psi y_T$$

for some *Lagrange multiplier*  $\psi \in \mathbb{R}$ . Since the constraints on the controls and states are convex and our cost function is strictly concave, Pontryagin's maximum principle asserts that the unique optimal solution  $q^*$  is given by

$$q_t^* = \arg \max_{\bar{q} \geq 0} H(x, \bar{q}, Y_t) = \frac{1}{2}(1 - Y_t)^+ \quad (2.2)$$

where the adjoint process  $Y$  satisfies the costate equation

$$\begin{aligned} -\dot{Y}_t &= -rY_t + \partial_x H(X_t^*, q_t^*, Y_t) \quad \text{for a.a. } t \in [0, T] \\ Y_T &= \partial_{y_T} L(Y_T^*, \psi^*) \end{aligned} \quad (2.3)$$

for some  $\psi^* \geq 0$ . In our setting the costate equation reduces to

$$-\dot{Y}_t = -rY_t, \quad Y_T = \psi^*$$

and solving the optimization problem reduces to finding the terminal value of the adjoint process.

Determining the terminal value of the adjoint process is equivalent to determining the value of the process at an arbitrary time  $t \in [0, T]$ . The canonical choice of time is the (candidate) optimal exhaustion time. To determine this time, for a given admissible control  $q$ , we denote by

$$\tau_q = \tau_q(x_0) := \inf \{t \in [0, T] : X_t^q = 0\}$$

the time of depletion of the resource under the control  $q$ . We put  $\tau_q = +\infty$  if  $X_T^q > 0$  denote by

$$\tau^* = \tau_{q^*}$$

the time of depletion under the candidate optimal control  $q^*$ . To determine the value  $Y_{\tau^*}$  we distinguish three cases depending on whether full exploitation occurs strictly before time  $T$ , exactly at the terminal time or whether full exploitation is not optimal.

If full exploitation occurs strictly before time  $T$  the optimization problem is equivalent to one on  $[0, \tau^*]$  and it follows from (2.2) that  $Y_{\tau^*} = 1$ . If exploitation does not occur ( $\tau^* = +\infty$ ), then the terminal state constraint is not binding and the revenue function can be maximized under the integral. The interesting case is the one where exploitation occurs exactly at time  $T$ . In this case, the terminal state of the adjoint process depends on the initial resource level. More precisely, we have the following:

- $\tau^* < T$ . In this case  $Y_{\tau^*} = 1$  to ensure that  $q^* \equiv 0$  in  $[\tau^*, T]$ . Thus

$$q_t^* = \frac{1}{2} \left(1 - e^{-r(\tau^*-t)}\right) \mathbb{1}_{\{t \leq \tau^*\}}$$

and  $\tau^*$  is determined by the identity

$$x_0 = \int_0^{\tau^*} q_t^* dt = \frac{1}{2} \int_0^{\tau^*} (1 - e^{-r(\tau^*-t)}) dt = \frac{\tau^*}{2} - \frac{1}{2r}(1 - e^{-r\tau^*}).$$

This gives

$$\tau^* = 2x_0 + \frac{1 + \mathbb{W}(-e^{-1-2rx_0})}{r}, \quad (2.4)$$

where  $\mathbb{W}$  denotes the principal branch of the Lambert-W function, defined as the inverse function of  $xe^x$ , restricted to the range  $[-1, \infty)$  and the domain  $[-e^{-1}, \infty)$ .<sup>4</sup>

- $\tau^* = T$ . In this case the terminal value  $Y_T$  of the adjoint process is determined through the equation

$$x_0 = \int_0^T q_t^* dt = \frac{1}{2} \int_0^T (1 - Y_T e^{-r(T-t)}) dt = \frac{T}{2} - \frac{Y_T}{2r}(1 - e^{-rT}),$$

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<sup>4</sup>This recovers the formula found by dynamic programming methods for the  $T = \infty$  case in [9, Proposition 3].

which yields

$$Y_T = \frac{r(T - 2x_0)}{1 - e^{-rT}}, \quad \text{and} \quad q_t^*(x_0) = \frac{1}{2} \left( 1 - \frac{r(T - 2x_0)}{1 - e^{-rT}} e^{-r(T-t)} \right). \quad (2.5)$$

Note that  $q_T^* > 0$  in general, and  $Y \geq 0$  only if  $x_0 \leq T/2$ . This shows that full exploitation is not optimal if  $x_0 > T/2$ .

- $\tau^* = +\infty$ . In this case,  $q^* \equiv \frac{1}{2}$  is admissible and hence optimal. Moreover,  $Y_t \equiv 0$  and necessarily  $x_0 \geq T/2$  (and so  $T < \infty$ ).

In terms of the above obtained values  $Y_{\tau^*(x_0)}$  the adjoint process as a function of the initial resource is given by

$$Y_t(x_0) = Y_{\tau^*(x_0)}(x_0) e^{-r(\tau^*(x_0)-t)}, \quad t \in [0, T]. \quad (2.6)$$

This process along with the control  $q^*$  defined in (2.2) satisfies the costate equation of Pontryagin's maximum principle. As a result,  $q^*$  is indeed optimal.

So far, we considered only finite horizon models. For infinite horizon problems it is in general not clear if Pontryagin's principle applies. In our setting the case  $T = +\infty$  is easier than the finite horizon case, though. If  $T = +\infty$ , then the optimization problem is equivalent to a problem on  $[0, \tau^*]$ . The following theorem summarizes our findings.

**Theorem 2.1.** *Let  $T \in (0, \infty]$ . The optimal control at time  $t \in [0, T]$  to the control problem (2.1) with initial state  $x_0 > 0$  is given by*

$$q_t^*(x_0) = \frac{1}{2} \begin{cases} (1 - e^{-r(\tau^*(x_0)-t)}) \mathbb{1}_{\{t \leq \tau^*(x_0)\}}, & 0 \leq x_0 \leq \frac{T}{2} - \frac{1}{2r}(1 - e^{-rT}); \\ 1 - Y_T(x_0) e^{-r(T-t)}, & \frac{T}{2} - \frac{1}{2r}(1 - e^{-rT}) \leq x_0 \leq T/2; \\ 1, & x_0 \geq T/2; \end{cases}$$

where  $\tau^*(x_0)$  and  $Y_T(x_0)$  are given by (2.4) and (2.5), respectively. In particular, the value function is given by the smooth function<sup>5</sup>

$$u(x_0) = \frac{1}{4r} \begin{cases} (1 + \mathbb{W}(-e^{-1-2rx_0}))^2, & 0 \leq x_0 \leq \frac{T}{2} - \frac{1-e^{-rT}}{2r}; \\ 1 - e^{-rT} - \frac{r^2(T-2x_0)^2}{e^{rT}-1}, & \frac{T}{2} - \frac{1-e^{-rT}}{2r} \leq x_0 \leq \frac{T}{2}; \\ 1 - e^{-rT}, & x_0 \geq \frac{T}{2}. \end{cases}$$

*Proof.* For notational convenience we drop the dependence of  $\tau^*$ ,  $Y_T$  and  $q^*$  on  $x_0$ . For any admissible control  $q$  with corresponding exhaustion time  $\tau^q$  the Hamiltonian satisfies

$$H(X_t^*, q_t^*, Y_t) \geq H(X_t^q, q_t, Y_t) \quad \text{on} \quad [0, \tau^* \wedge \tau^q]. \quad (2.7)$$

and because  $Y_t \geq 1$  for  $t \geq \tau^*$  it also satisfies

$$H(X_t^q, q_t, Y_t) \leq 0 \quad \text{on} \quad [\tau^*, T]. \quad (2.8)$$

We now put  $\bar{\tau}_q := \tau_q \wedge T$  and  $\bar{\tau}^* := \tau^* \wedge T$  and distinguish two cases.

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<sup>5</sup>This corresponds to the solution to the dynamic programming equation in the  $T = \infty$  case found in [22, Section 5].

- $\tau^* \geq \tau_q$ . It follows from (2.7) along with the fact that the Hamiltonian is non-negative along the candidate optimal solution and because  $Y_t e^{-rt} = Y_0$  for all  $t \in [0, T]$  that

$$\begin{aligned}
J(q^*) - J(q) &= \int_0^{\bar{\tau}^*} e^{-rt} q_t^* (1 - q_t^*) dt - \int_0^{\bar{\tau}_q} e^{-rt} q_t (1 - q_t) dt \\
&= \int_0^{\bar{\tau}_q} e^{-rt} (H(X_t^*, q_t^*, Y_t) - H(X_t^q, q_t, Y_t) + (q_t^* - q_t) Y_t) dt \\
&\quad + \int_{\bar{\tau}_q}^{\bar{\tau}^*} e^{-rt} (H(X_t^*, q_t^*, Y_t) + q_t^* Y_t) dt \\
&\geq \int_0^{\bar{\tau}_q} (q_t^* - q_t) Y_t e^{-rt} dt + \int_{\bar{\tau}_q}^{\bar{\tau}^*} q_t^* Y_t e^{-rt} dt \\
&= Y_0 \left( \int_0^{\bar{\tau}^*} q_t^* dt - \int_0^{\bar{\tau}_q} q_t dt \right) \\
&= 0
\end{aligned}$$

where the last equality follows from the fact that  $Y_0 = 0$  if  $\tau^* > T$  while the term in parenthesis vanishes if  $\tau^* \leq T$  in which case both integrals equal  $x_0$ .

- The case  $\tau^* < \tau_q$ . In this case, it follows again from (2.7) and that

$$\begin{aligned}
J(q^*) - J(q) &= \int_0^{\bar{\tau}^*} e^{-rt} (H(X_t^*, q_t^*, Y_t) - H(X_t^q, q_t, Y_t) + (q_t^* - q_t) Y_t) dt \\
&\quad - \int_{\bar{\tau}^*}^{\bar{\tau}_q} e^{-rt} (H(X_t, q_t, Y_t) + q_t Y_t) dt \\
&\geq \int_0^{\bar{\tau}^*} (q_t^* - q_t) Y_t e^{-rt} dt - \int_{\bar{\tau}^*}^{\bar{\tau}_q} e^{-rt} (H(X_t^q, q_t, Y_t) + q_t Y_t) dt \\
&\geq Y_0 \left( \int_0^{\bar{\tau}^*} q_t^* dt - \int_0^{\bar{\tau}_q} q_t dt \right).
\end{aligned}$$

If  $\tau^* > T$ , then  $Y_0 = 0$ . Else,  $Y_0 \geq 0$ ,  $\int_0^{\bar{\tau}^*} q_t^* dt = x_0$  and  $\int_0^{\bar{\tau}_q} q_t dt = x_0 - X_{\bar{\tau}_q}^q$ . As a result,

$$J(q^*) - J(q) \geq 0.$$

□

### 3 The Mean Field Game

To motivate the form of demand functions that we are going to use in the continuum MFG, we first introduce a finite market with  $N$  oil producers that compete for market share in a one-period game. Associated to each firm  $i \in \{1, \dots, N\}$  are variables  $p_i \in \mathbb{R}$  and  $q_i \in \mathbb{R}_+$  representing the price and quantity, respectively. In the Cournot model, players choose quantities as a strategic variable in non-cooperative competition with the other firms, and the market determines the price of each good. The market model is specified by linear inverse demand functions, which give prices

as a function of quantity produced. The firms are suppliers, and so quantities are nonnegative. For  $q \in \mathbb{R}_+^N$ , the price received by player  $i$  is  $p_i = P_i(q)$  where

$$P_i(q) = 1 - (q_i + \epsilon \bar{q}_i), \quad \text{where} \quad \bar{q}_i = \frac{1}{N-1} \sum_{j \neq i} q_j, \quad i = 1, \dots, N, \quad \text{and} \quad 0 \leq \epsilon < N-1. \quad (3.1)$$

The inverse demand functions are decreasing in all of the quantities, and  $\epsilon$  measures the strength of interaction between players. In the linear model (3.1), some of the prices  $p_i = P_i(q)$  may be negative, meaning player  $i$  produces so much that he has to pay to have his goods taken away, but negative prices do not arise in competitive equilibrium. Moreover, the goods are similar but differentiated, meaning each player potentially receives a different price as there will be some residual loyalty to obtaining the product from individual suppliers. Most crucially, the interaction is of mean field type:  $p_i$  is affected by the mean production of the other players, and players  $j$  and  $k$  ( $j, k \neq i$ ) are exchangeable as far as player  $i$  is concerned.

In the MFG version, there is a continuum of players, say oil producers, who are labeled by their reserves at time. Initial reserves are distributed according to the probability measure  $\mu$  on  $[0, \infty)$ . Each producer extracts oil in continuous time at a rate  $q_t \geq 0$ , and the price received by this producer is  $P(q_t, Q_t) = 1 - q_t - \epsilon Q_t$ , where  $Q_t$  is the mean production rate of all the players, and  $\epsilon \geq 0$  quantifies the degree of interaction between the players. In the following, the time horizon  $T \leq \infty$ .

### 3.1 The best response function

In a first step, we consider the representative player's best response *given* the aggregate production of all the other players. Aggregate production is described by a nonnegative and absolutely continuous function  $Q : [0, T] \rightarrow [0, \frac{1}{2+\epsilon}]$ . The derivative of  $Q$  exists a.e. and is denoted by  $\dot{Q}$ .

*Remark 3.1.* It is in fact sufficient here to simply assume  $Q \leq 1/\epsilon$  which is immediately evident to guarantee nonnegative prices. The a priori bounds  $0 \leq Q \leq 1/(2 + \epsilon)$  are motivated by the following observation: for any candidate optimal control we have

$$0 \leq q \leq \arg \max_{q \geq 0} \{q(1 - \epsilon Q - q)\} = \frac{1}{2}(1 - \epsilon Q)^+,$$

where the right-hand side is the optimal control given infinite reserves. Any aggregate production function  $Q$  leading to a solution to the MFG therefore necessarily satisfies

$$0 \leq Q \leq \frac{1}{2}(1 - \epsilon Q) \quad \text{and so} \quad Q \leq \frac{1}{2 + \epsilon}.$$

We assume throughout that the aggregate production function satisfies the following compatibility condition. We will see that this condition guarantees that each player fully exploits her initial resources if  $T$  is large enough. The assumption will be satisfied in equilibrium.

**Assumption 1.** There exists  $\delta > 0$  such that the aggregate exploitation rate  $Q : [0, T] \rightarrow [0, \frac{1}{2+\epsilon}]$  satisfies the compatibility condition

$$1 - \epsilon Q + \frac{\epsilon}{r} \dot{Q} \geq \delta > 0. \quad (3.2)$$



*Remark 3.2.* On finite time intervals our compatibility condition is equivalent to  $\frac{d}{dt}(e^{-rt}Q_t) > \frac{1}{\varepsilon} \frac{d}{dt}(e^{-rt})$ . Since we expect  $Q$  to be decreasing in equilibrium (production slows as resources run out), the condition (3.2) puts a lower bound on how quickly that may occur over time.

Let us now consider a representative producer with any initial state  $x_0 \in [0, \infty)$  at time 0. As in the case of a monopolist producer we call a control  $q : [0, T] \rightarrow [0, 1]$  admissible if the state process

$$X_t^q := x_0 - \int_0^t q_u du$$

is always non-negative. For  $T < \infty$  or  $T = \infty$ , depending on whether the finite or infinite horizon case is considered, the value function for the representative producer with respect to a given function  $Q$  is defined by

$$u^Q(x_0) = \sup_{q \geq 0} J^Q(q), \quad \text{where} \quad J^Q(q) := \int_0^{\tau_q^Q \wedge T} e^{-rt} q_t (1 - \varepsilon Q_t - q_t) dt, \quad (3.3)$$

and the exhaustion time  $\tau_q^Q = \tau_q^Q(x_0)$  is

$$\tau_q^Q := \inf \{t \in [0, T] \mid X_t^q = 0\}.$$

By analogy to the single player case, the Hamiltonian and the end-point Lagrangian are given by, respectively,

$$\begin{aligned} H(x, q, y) &= q(1 - \varepsilon Q - q) - qy \\ L(y_T, \psi) &= \psi y_T. \end{aligned} \quad (3.4)$$

As in the single player case Pontryagin's maximum principle asserts that a control  $q^Q$  is optimal control, with associated *optimal* state process  $X^{q^Q}$  and vanishing time  $\tau^Q$ , if  $q^Q$  satisfies the maximum condition

$$q_t^Q = \arg \max_{\bar{q} \geq 0} H(X_t^{q^Q}, \bar{q}, Y_t) = \frac{1}{2} (1 - \varepsilon Q_t - Y_t)^+ \quad (3.5)$$

and the adjoint process satisfies

$$Y_t^Q = Y_{\tau^Q}^Q e^{-r(\tau^Q - t)}, \quad t \geq 0 \quad (3.6)$$

where value of the adjoint process at time  $\tau^Q$  is again to be determined. We distinguish again three different cases depending on whether full exploitation is optimal or not.

- $\tau^Q < T \leq \infty$ . In this case  $q_{\tau^Q}^Q = 0$  and we infer from (3.5) that  $Y_{\tau^Q}^Q = 1 - \varepsilon Q_{\tau^Q}$ . Hence, from (3.6), we have

$$Y_t^Q = (1 - \varepsilon Q_{\tau^Q}) e^{-r(\tau^Q - t)}, \quad t < \tau^Q,$$

and the optimal strategy is

$$q_t^Q(x_0) = \frac{1}{2} \left( 1 - \varepsilon Q_t - (1 - \varepsilon Q_{\tau^Q(x_0)}) e^{-r(\tau^Q(x_0) - t)} \right), \quad t < \tau^Q. \quad (3.7)$$

- $\tau^Q = T < \infty$ . In this case the terminal value  $Y_T^Q$  of the adjoint process is obtained by the identity

$$x_0 = \int_0^T q_t^Q(x_0) dt = \frac{1}{2} \int_0^T (1 - \varepsilon Q_t - Y_T(x_0)e^{-r(T-t)}) dt,$$

which yields

$$Y_T^Q = \frac{1}{\beta_T} \left( \frac{1}{2} \int_0^T (1 - \varepsilon Q_t) dt - x_0 \right), \quad \text{where } \beta_T := \frac{1 - e^{-rT}}{2r} \quad (3.8)$$

and the optimal strategy is given by

$$q_t^Q(x_0) = \frac{1}{2} (1 - \varepsilon Q_t - Y_T(x_0)e^{-r(T-t)}), \quad t < T.$$

We notice that the adjoint process is non-negative if and only if

$$x_0 \leq \eta^Q(T), \quad \text{where } \eta^Q(T) := \frac{1}{2} \int_0^T (1 - \varepsilon Q_t) dt. \quad (3.9)$$

- $T < \infty$  and  $\tau^Q > T$ . In this case the strategy

$$q_t^Q = \frac{1}{2} (1 - \varepsilon Q_t).$$

is admissible and hence optimal and the adjoint process is identically equal to zero,  $Y \equiv 0$ . This case occurs when  $x_0 \geq \eta^Q(T)$ .

Having determined the full dynamics of the adjoint process and hence the optimal control is remains to characterize the optimal exhaustion time. To this end, it will be convenient to introduce the function

$$\xi^Q(t) := \frac{1}{2} \int_0^t \{1 - \varepsilon Q_s - (1 - \varepsilon Q_t)e^{-r(t-s)}\} ds, \quad (3.10)$$

that specifies is the amount of resource extracted up to time  $t$  by using the strategy (3.7) that would be optimal if the exhaustion time  $\tau^Q$  was equal to  $t$ . Its derivative is given by

$$\dot{\xi}^Q(t) = \frac{1}{2} \left\{ 1 - \varepsilon Q_t + \frac{\varepsilon}{r} \dot{Q}_t \right\} (1 - e^{-rt}). \quad (3.11)$$

Assumption 1 guarantees that  $\xi^Q(\infty) = \infty$  and that  $\dot{\xi}^Q(t) > 0$ . As a result, the inverse  $(\xi^Q)^{-1}$  is well-defined on  $[0, \xi^Q(T)]$ . Moreover, we see that

$$\xi^Q(\tau^Q(x_0)) = \int_0^{\tau^Q(x_0)} q_s^Q(x_0) ds, \quad (3.12)$$

and hence that  $\tau^Q(x_0)$  is determined implicitly by the identity

$$\xi^Q(\tau^Q(x_0)) = x_0, \quad \text{i.e. } \tau^Q(x_0) = (\xi^Q)^{-1}(x_0) \quad \text{for } x_0 \leq \xi^Q(T).$$

Thus, an optimal exploitation strategy for a given initial resource level  $x_0$  and a given aggregate production function  $Q$  that satisfies our compatibility condition can be obtained by first computing the optimal exploitation time by inverting the function  $\xi^Q$  introduced in (3.10) on its range  $[0, \xi^Q(T)]$ , and then applying the maximum principle on  $[0, \tau^Q(x_0)]$ . The following theorem summarizes our findings. The proof is the same as in the single player case and is hence omitted.

**Theorem 3.3.** Let  $Q : [0, T] \rightarrow [0, \frac{1}{2+\varepsilon}]$  be an absolutely continuous function that satisfies the compatibility condition (3.2). Define the function  $\xi^Q : [0, T] \rightarrow [0, \xi^Q(T)]$  by (3.10), and let  $\tau^Q = (\xi^Q)^{-1}$ .

- $T = \infty$ . For initial state  $x_0 \in [0, \infty)$ ,  $\tau^Q(x_0) < \infty$ , and the optimal control is given by

$$q_t^Q(x_0) = \frac{1}{2} \{1 - \varepsilon Q_t - (1 - \varepsilon Q_{\tau^Q(x_0)}) e^{-r(\tau^Q(x_0) - t)}\} \mathbb{1}_{\{t \leq \tau^Q(x_0)\}}. \quad (3.13)$$

- $T < \infty$ . The optimal control for the initial state  $x_0 \in [0, \infty)$  is given by

$$q_t^Q(x_0) = \begin{cases} \frac{1}{2} \{1 - \varepsilon Q_t - (1 - \varepsilon Q_{\tau^Q(x_0)}) e^{-r(\tau^Q(x_0) - t)}\} \mathbb{1}_{\{t \leq \tau^Q(x_0)\}}, & x_0 \in [0, \xi^Q(T)] \\ \frac{1}{2} (1 - \varepsilon Q_t - Y_T(x_0) e^{-r(T-t)}) & x_0 \in [\xi^Q(T), \eta^Q(T)], \\ \frac{1}{2} (1 - \varepsilon Q_t), & \text{else,} \end{cases} \quad (3.14)$$

where  $Y_T(x_0)$  and  $\eta^Q(T)$  are defined in (3.8) and (3.9), respectively.

It follows from the above theorem that all the optimal production rates  $q_t^Q$  satisfy the same ODE, albeit on possibly different time horizons and with possibly different terminal conditions.

**Corollary 3.4.** For all  $x_0 > 0$  and all functions  $Q$  that satisfy the compatibility condition (3.2), the optimal rate  $q_t^Q(x_0)$  satisfies the ODE

$$q_t^Q(x_0) - \frac{1}{r} \dot{q}_t^Q(x_0) = \frac{1}{2} \left\{ 1 - \varepsilon Q_t + \frac{\varepsilon}{r} \dot{Q}_t \right\} \quad \text{on } (0, \tau^Q(x_0) \wedge T). \quad (3.15)$$

In the case  $T = +\infty$ , the terminal condition is  $q_{\tau^Q(x_0)}^Q(x_0) = 0$ . If  $T < \infty$ , then the terminal condition is given in terms of the  $Y_T(x_0)$  given in (3.8) as

$$q_{\tau^Q(x_0) \wedge T}^Q(x_0) = \begin{cases} 0 & \text{if } x_0 \in [0, \xi^Q(T)] \\ \frac{1}{2} (1 - \varepsilon Q_T - Y_T(x_0)) & \text{if } x_0 \in [\xi^Q(T), \eta^Q(T)] \\ \frac{1}{2} (1 - \varepsilon Q_T) & \text{else,} \end{cases} \quad (3.16)$$

*Proof.* The ODE (3.15) follows from differentiating the equation  $q_t^Q = \frac{1}{2}(1 - \varepsilon Q_t - Y_t)^+$  for the optimal production rate, noticing that the positive part can be dropped because  $t \leq \tau^Q(x_0) \wedge T$ , using the ODE for  $Y$ , and substituting back for  $Y$  in terms of  $Q$  and  $q^Q$ .  $\square$

Next, we consider the dynamics on the aggregate production rate under the following assumption on the initial distribution of states.

**Assumption 2.** The initial distribution  $\mu$  is of the form

$$\mu(dx) = f(x) dx + \sum_{i=1}^n c_i \delta_{x_i}(dx)$$

for a bounded non-negative function  $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  and non-negative constants  $c_1, \dots, c_n$ . Here  $\delta_x$  denotes the Dirac measure on  $\{x\}$ .

From equation (3.11) we see that the mapping  $x_0 \mapsto \tau^Q(x_0)$  is strictly increasing and so

$$\{x_0 \geq 0 : \tau^Q(x_0) < t\} = \{x_0 \geq 0 : x_0 < \xi^Q(t)\}.$$

In particular, if we denote by

$$S_\mu(x) := \mu((x, \infty))$$

the survival function associated with the initial distribution  $\mu$ , then the proportion of players with remaining resources at time  $t \in [0, T]$  is given by  $S_\mu(\xi^Q(t))$ . This allows to obtain the dynamics of the aggregate production rate

**Lemma 3.5.** *For all  $x_0 > 0$  and all bounded, absolutely continuous functions  $Q$  that satisfy the compatibility condition (3.2), the aggregate production rate*

$$Q_t^Q := \int_0^\infty q_t^Q(x_0) d\mu(x_0), \quad 0 \leq t \leq T \quad (3.17)$$

is bounded and absolutely continuous, and its density  $\dot{Q}_t^Q$  satisfies the ODE

$$Q_t^Q - \frac{1}{r} \dot{Q}_t^Q = \frac{1}{2} \left\{ 1 - \varepsilon Q_t + \frac{\varepsilon}{r} \dot{Q}_t \right\} S_\mu(\xi^Q(t)) \quad \text{for a.a. } t \in (0, T). \quad (3.18)$$

In the infinite horizon case, the terminal condition is  $\lim_{t \rightarrow \infty} Q_t = 0$ . If  $T < \infty$ , then

$$\begin{aligned} Q_T^Q &= \frac{1}{2} (1 - \varepsilon Q_T) S_\mu(\xi^Q(T)) - \frac{\eta^Q(T)}{\beta_T} (S_\mu(\xi^Q(T)) - S_\mu(\eta^Q(T))) \\ &\quad + \frac{1}{\beta_T} \int_{\xi^Q(T)}^{\eta^Q(T)} x \mu(dx). \end{aligned} \quad (3.19)$$

*Proof.* Since  $q^Q$  is bounded, due to the boundedness of  $Q$ , the aggregate production rate is bounded. Since  $q_t^Q(x) = 0$  for all  $t \in [\tau^Q(x), \infty)$  the aggregate production rate can be expressed as

$$Q_t^Q = \int_0^\infty \mathbb{1}_{(\xi_t^Q, \infty)}(x) q_t^Q(x) \mu(dx)$$

By Fubini's theorem

$$\begin{aligned} Q_t^Q - Q_0^Q &= \int_0^\infty (q_t^Q(x) - q_0^Q(x)) \mu(dx) \\ &= \int_0^\infty \int_0^t \dot{q}_s^Q(x) ds \mu(dx) \\ &= \int_0^t \int_0^\infty \dot{q}_s^Q(x) \mu(dx) ds. \end{aligned}$$

This shows that  $Q^Q$  is absolutely continuous with derivative  $\dot{Q}_t = \int_0^\infty \dot{q}_t^Q(x) \mu(dx)$ . Using that

$$\dot{q}_t^Q(x) = 0 \quad \text{for } t \in (\tau^Q(x), T)$$

(note that  $\dot{q}_{\tau^Q(x)}^Q(x)$  may be positive) and that the mapping  $t \mapsto \xi_t$  is strictly increasing the aggregate marginal rate of production can equivalently be expressed as

$$\dot{Q}_t^Q = \int_0^\infty \mathbb{1}_{[\xi_t^Q, \infty)}(x) \dot{q}_t^Q(x) \mu(dx) = \int_0^\infty \mathbb{1}_{(\xi_t^Q, \infty)}(x) \dot{q}_t^Q(x) \mu(dx) \quad \text{a.e.}$$

In view of Corollary 3.4 this proves (3.18). The terminal condition on  $Q^Q$  is obtained from integrating (3.16) with respect to  $\mu$  over  $(0, \infty)$ . In view of Assumption 2 the ODE (3.18) can be solved backwards in time.  $\square$

*Remark 3.6.* Under the compatibility condition, we have  $\xi^Q(T) \uparrow \infty$  as  $T \uparrow \infty$ . As a result,  $\lim_{T \rightarrow \infty} Q_T^Q = 0$ . This shows that the infinite horizon case can indeed be viewed as a limiting case when  $T \uparrow \infty$ .

### 3.2 Solution to the MFG

In this section we prove the existence of a solution to the mean field game. In terms of the optimal controls  $q^Q$  we are looking for a fixed point of the mapping (3.17). We consider the infinite and the finite horizon case separately. The main difference is that full exploitation occurs in equilibrium if  $T = +\infty$ , but may not occur if  $T < \infty$ . It turns out that the infinite horizon case can be solved in closed form. In view of Corollary 3.4, we have the following characterization.

**Corollary 3.7.** *Any fixed point  $Q^* = Q^{Q^*}$  satisfies the dynamics*

$$Q^* - \frac{1}{r} \dot{Q}^* = \frac{S_\mu \circ \xi^{Q^*}}{2 + \varepsilon S_\mu \circ \xi^{Q^*}},$$

and satisfies the compatibility condition (3.2).

*Proof.* Setting  $Q^Q$  and  $Q$  in (3.18) to  $Q^*$  and re-arranging terms leads to the above equation, which is equivalent to

$$1 - \varepsilon Q^* + \frac{\varepsilon}{r} \dot{Q}^* = \frac{2}{2 + \varepsilon S_\mu \circ \xi^{Q^*}}. \quad (3.20)$$

From this, compatibility of  $Q^*$  easily follows.  $\square$

For any fixed point  $Q^*$  the equations (3.11) and (3.20) yield the following ODE for  $\xi^{Q^*}$ :

$$\dot{\xi}^{Q^*}(t) = \frac{1 - e^{-rt}}{2 + \varepsilon S_\mu \circ \xi^{Q^*}}, \quad \xi_t^{Q^*}(0) = 0.$$

The key observation is that (3.21) does not depend on  $Q^*$ . We prove below that the (unique) solution to the MFG can be defined in terms of the unique solution  $\xi^*$  to the above ODE.

**Proposition 3.8.** *Under Assumption 2 the ODE*

$$\dot{\xi}^*(t) = \frac{1 - e^{-rt}}{2 + \varepsilon S_\mu \circ \xi^*}, \quad \xi_t^*(0) = 0. \quad (3.21)$$

has a unique global solution. The solution can be expressed in terms of the functions  $\phi, \psi : [0, \infty) \rightarrow [0, \infty)$  defined by

$$\phi(t) := \frac{rt - 1 + e^{-rt}}{r} \quad \text{and} \quad \psi(x) := 2x + \varepsilon \int_0^x S_\mu(\xi) d\xi,$$

respectively as

$$\xi^* := \psi^{-1} \circ \phi \tag{3.22}$$

*Proof.* If  $\mu$  is absolutely continuous with bounded density, then the mapping  $x \mapsto \frac{1-e^{-rt}}{2+\varepsilon S_\mu(x)}$  is Lipschitz continuous. If  $\mu$  has an additional jump part, then this mapping is piecewise Lipschitz continuous and hence can be solved iteratively. The functions  $\phi$  and  $\psi$  are well-defined, strictly increasing, surjective, and satisfy

$$\dot{\phi}(t) = 1 - e^{-rt} \quad \text{and} \quad \psi'(x) = 2 + \varepsilon S_\mu(x).$$

Now, a direct computation verifies that  $\xi^*$  satisfies the desired ODE.  $\square$

*Remark 3.9.* Note that the inverse of  $\phi$  is, in terms of the principle branch of the Lambert-W function,

$$\phi^{-1}(x) = x + \frac{1 + \mathbb{W}(-e^{-1-rx})}{r}.$$

### 3.2.1 Infinite horizon

We are now ready to solve the MFG in the infinite horizon case. In terms of the functions  $\phi$  and  $\psi$ , and their inverses we obtain

- $\xi^* := \psi^{-1} \circ \phi$
- $\tau^* := \xi^{*-1} = \phi^{-1} \circ \psi$

In view of (3.20) we can express the unique solution to the ODE (3.15) with terminal condition (3.18) fully in terms of these functions as

$$q_t^*(x_0) := \int_{t \wedge \tau^*(x_0)}^{\tau^*(x_0)} \frac{r e^{-r(s-t)}}{2 + \varepsilon S_\mu(\xi^*(s))} ds. \tag{3.23}$$

The following theorem verifies that this production rate does indeed solve the MFG in the infinite horizon case.

**Theorem 3.10.** *The aggregate production rate*

$$Q_t^* := \int_0^\infty q_t^*(x_0) d\mu(x_0)$$

*is absolutely continuous, takes values in  $[0, \frac{1}{2+\varepsilon}]$  and satisfies the compatibility condition (3.2) as well as the fixed point property  $Q^{Q^*} = Q^*$ . Hence,  $Q^*$  is the unique solution to the MFG (that is absolutely continuous and satisfies the compatibility condition).*

*Proof.* Let us first verify that  $Q^*$  satisfies the compatibility condition. The control  $q^*$  defined in (3.23) satisfies

$$q_t^*(x_0) - \frac{1}{r} \dot{q}_t^*(x_0) = \frac{1}{2 + \varepsilon S_\mu(\xi^*(t))} \mathbb{1}_{\{t < \tau^*(x_0)\}}. \quad (3.24)$$

By analogy to Lemma 3.5 this yields that  $Q^*$  satisfies the ODE

$$Q_t^* - \frac{1}{r} \dot{Q}_t^* = \frac{S_\mu(\xi^*(t))}{2 + \varepsilon S_\mu(\xi^*(t))} \quad (3.25)$$

with terminal condition  $\lim_{t \rightarrow \infty} Q_t^* = 0$ . In particular,

$$Q_t^* = \int_t^\infty \frac{r S_\mu(\xi^*(s)) e^{-r(s-t)}}{2 + \varepsilon S_\mu(\xi^*(s))} ds. \quad (3.26)$$

Using that

$$\frac{S_\mu(\xi^*(t))}{2 + \varepsilon S_\mu(\xi^*(t))} \leq \frac{1}{2 + \varepsilon}$$

we conclude that  $0 \leq Q_t^* \leq \frac{1}{2 + \varepsilon}$  and hence that  $Q^*$  satisfies the compatibility condition.

To verify the fixed point property  $Q^{Q^*} = Q^*$  we first notice that (3.25) is equivalent to

$$\frac{1}{2} \left\{ 1 - \varepsilon Q_t^* + \frac{\varepsilon}{r} \dot{Q}_t^* \right\} = \frac{1}{2 + \varepsilon S_\mu(\xi^*(t))}.$$

In view of (3.11) and because  $\xi^*$  satisfies the ODE (3.21) we conclude that

$$\xi^* = \xi^{Q^*} \quad \text{and hence} \quad \tau^* = \tau^{Q^*}.$$

This shows that  $q^{Q^*} = q^*$  as they solve the same ODE, and hence the fixed point property  $Q^{Q^*} = Q^*$ . An application of Theorem 3.3 verifies that  $Q^*$  is the unique solution the MFG.  $\square$

While the individual production rate  $q^*$  may not be monotone in general as illustrated below, it is a direct consequence of (3.26) and (3.25) that the aggregate production rate is nonincreasing.

**Corollary 3.11.** *The aggregate production rate  $Q^*$  is nonincreasing.*

### 3.2.2 Finite horizon

In the finite horizon case an additional challenge emerges. While we can still solve the ODE (3.15) using (3.20), due to the dependence of the second and third terminal conditions in (3.16) on  $Q_T$  we obtain the equilibrium production rate only up to its terminal value.<sup>6</sup> An additional fixed point argument on the terminal value  $Q_T^*$  is required to solve the game. In terms of the yet to be determined terminal condition  $Q_T^*$  we do know that

$$Q_t^* = Q_T^* e^{-r(T-t)} + \int_t^T \frac{r S_\mu(\xi^*(s)) e^{-r(s-t)}}{2 + \varepsilon S_\mu(\xi^*(s))} ds.$$

---

<sup>6</sup>In the infinite horizon game both the dynamics and the terminal conditions were defined in terms of  $\xi^*$  and its inverse  $\tau^*$ . This is no longer the case in the finite horizon game.

In terms of

$$\psi^Q(x_0) := \frac{1}{\beta_T} (x_0 - \xi^Q(T)) \quad (3.27)$$

we obtain for any admissible process  $Q$ ,

$$\begin{aligned} Q_T^Q &= \frac{1}{2} \int_{\xi^Q(T)}^{\xi^Q(T) + \beta_T(1 - \varepsilon Q_T)} \psi^Q(x_0) d\mu(x_0) + \frac{1}{2} \int_{\xi^Q(T) + \beta_T(1 - \varepsilon Q_T)}^{\infty} (1 - \varepsilon Q_T^Q) d\mu(x_0) \\ &= \frac{1}{2} \int_{\xi^Q(T)}^{\infty} \{\psi^Q(x_0) \wedge (1 - \varepsilon Q_T^Q)\} d\mu(x_0). \end{aligned}$$

Putting

$$\psi^*(x_0) := \frac{1}{\beta_T} (x_0 - \xi^*(T))$$

we obtain for any aggregate equilibrium production rate function  $Q^*$  that

$$Q_T^* = \frac{1}{2} \int_{\xi^*(T)}^{\infty} \{\psi^*(x_0) \wedge (1 - \varepsilon Q_T^*)\} d\mu(x_0).$$

This means that  $Q_T^*$  is a fixed point of the map  $\Gamma : [0, \frac{1}{2}] \rightarrow [0, \frac{1}{2}]$  defined by

$$\Gamma(Q) = \frac{1}{2} \int_{\xi^*(T)}^{\infty} \{(\psi^*(x_0) \wedge \{1 - \varepsilon Q\}) \vee 0\} d\mu(x_0),$$

where we added the nonnegative cut-off (being redundant for any fixed point) to obtain a map on  $[0, \frac{1}{2}]$ . Moreover,  $\Gamma$  is strictly decreasing and continuous. Hence, it admits a unique fixed point  $Q^* = \Gamma(Q^*)$ . Since

$$\Gamma(Q^*) \leq \frac{1}{2} S_\mu(\xi^*(T))(1 - \varepsilon Q^*),$$

it follows that

$$Q^* \leq \frac{S_\mu(\xi^*(T))}{2 + \varepsilon S_\mu(\xi^*(T))}.$$

This is important to verify that the induced  $Q^*$  takes indeed values in  $[0, \frac{1}{2 + \varepsilon}]$ . For  $t \in [0, T]$ ,

$$\begin{aligned} Q_t^* &= Q^* e^{-r(T-t)} + \int_t^T \frac{r S_\mu(\xi^*(s)) e^{-r(s-t)}}{2 + \varepsilon S_\mu(\xi^*(s))} ds \\ &\leq \frac{S_\mu(\xi^*(t))}{2 + \varepsilon S_\mu(\xi^*(t))} e^{-r(T-t)} + \frac{S_\mu(\xi^*(t))}{2 + \varepsilon S_\mu(\xi^*(t))} \int_t^T r e^{-r(s-t)} ds \\ &= \frac{S_\mu(\xi^*(t))}{2 + \varepsilon S_\mu(\xi^*(t))}. \end{aligned}$$

The fact that  $Q^*$  satisfies the compatibility condition has already been established in Corollary 3.7. Let us summarize.

**Theorem 3.12.** *In terms of the above definitions, the function  $Q^*$  is absolutely continuous, takes values in  $[0, \frac{1}{2 + \varepsilon}]$ , and satisfies the compatibility condition (3.2) and the fixed point property*

$$Q^{Q^*} = Q^*.$$

*Hence,  $Q^*$  is the unique solution to the finite-time MFG (that is absolutely continuous and satisfies the compatibility condition).*



### 3.3 Examples

We close this section with two examples with infinite time horizon that illustrate that equilibrium production rates need not be monotone, nor smooth.

#### 3.3.1 Discrete initial distribution

Let us assume that the population of producers splits into two groups within which producers are identical. Producers in Group 1 have an initial resource  $x_1$ ; producers in Group 2 have an initial resource  $x_2$ . That is, the initial distribution is given by

$$\mu = (1 - p_2)\delta_{x_1} + p_2\delta_{x_2}$$

for  $0 \leq x_1 \leq x_2 < \infty$  and  $0 \leq p_2 \leq 1$  where  $p_2$  denotes the proportion of producers that belong to Group 2. We refer to a representative producer in Group  $i$  as Player  $i$  where  $i = 1, 2$ . The equilibrium production rates can be computed in closed form. The equilibrium production rate of Player 1 is given by

$$q_t^*(x_1) = \frac{1 - e^{-r(\tau_1-t)}}{2 + \varepsilon} \mathbb{1}_{\{t \leq \tau_1\}}, \quad (3.28)$$

where  $\tau_1 := \phi^{-1}((2 + \varepsilon)x_1)$ , while the equilibrium production rate of Player 2 is given by

$$q_t^*(x_2) = \begin{cases} \frac{1 - e^{-r(\tau_1-t)}}{2 + \varepsilon} + \frac{e^{-r(\tau_1-t)} - e^{-r(\tau_2-t)}}{2 + \varepsilon p_2}, & t \leq \tau_1, \\ \frac{1 - e^{-r(\tau_2-t)}}{2 + \varepsilon p_2} \mathbb{1}_{\{t \leq \tau_2\}}, & t \geq \tau_1, \end{cases}$$

where  $\tau_2 := \phi^{-1}(\varepsilon(1 - p_2)x_1 + (2 + \varepsilon p_2)x_2)$ . We notice that the equilibrium rate of Player 2 is continuous, but not differentiable at time  $\tau_1$ . Figure 1 illustrates the equilibrium production rates  $q_t^*(x_i)$  and resources  $X_t^*(x_i)$  of both players. While both Players initially produce at the same rate, Player 1 produces at a decreasing rate while Player 2 initially produces at an increasing rate, and then at a decreasing rate once Player 1 has run out of resources. Player 2 anticipates the fact Player 1 will eventually drop out of the market; her production rate reaches it peak at the time Player 1's resources have been depleted.

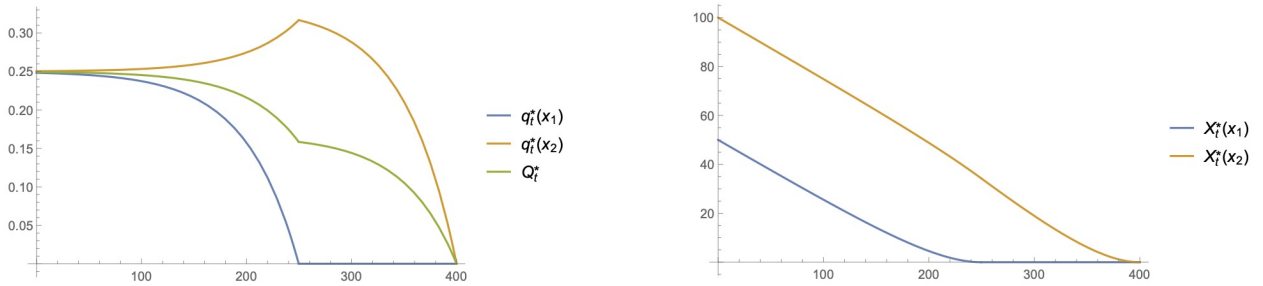


Figure 1:  $r = .02$ ,  $\varepsilon = 2$ ,  $x_1 = 50$ ,  $x_2 = 100$ ,  $p_2 = .5$

### 3.3.2 Exponential initial distribution

Figure 2 illustrates the equilibrium production rates  $q_t^*(i)$  for players with initial resources  $x_i = 1, 2, 3$  for the benchmark case of an exponential initial distribution. For absolutely continuous initial distributions players “gradually” drop out of the market so equilibrium production rates do not display kinks. However, we still observe non-monotonicity of production rates. Players with larger resources do again anticipate that players with lower reserves will eventually drop out of the game. For them it is optimal to produce at higher rates once the number of and the price pressure from their competitors decreases.

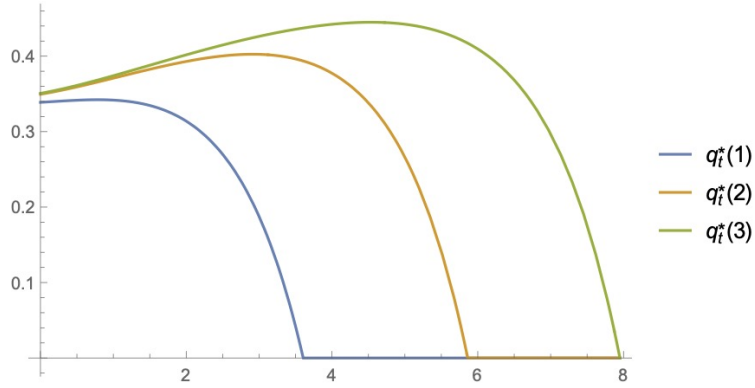


Figure 2:  $\mu \sim \text{Exp}(\lambda)$ ,  $r = \varepsilon = \lambda = 1$

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