Multi-Dimensional Mean Field Games with Singular Controls

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Abstract

This note fills a small gap of dimensionality in our early work [1]. In [1] we only focus on the one-dimensional model. In fact, the extension to multi-dimensional case is straightforward by introducing the Skorokhod weak M_1 topology.[1]

1 Multi-dimensional MFGs with singular controls

In [1] we prove the existence of equilibrium of one-dimensional MFGs with singular controls by introducing Skorokhod *strong* M_1 topology (SM_1). In [1] we claim the oscillation function for SM_1 disappears for monotone $z \in \mathcal{D}([0,T])$, using the fact

$$|z_{t_2} - [z_{t_1}, z_{t_3}]| = 0 \text{ for any } t_1 < t_2 < t_3 \tag{1.1}$$

where $|z_{t_2} - [z_{t_1}, z_{t_3}]|$ is defined as the distance from z_{t_2} to the segment $[z_{t_1}, z_{t_3}]$. However, (1.1) is only true if z is one-dimensional. For the multi-dimensional case, one can instead work with the Skorokhod *weak* M_1 topology (WM_1). For one dimensional trajectories, SM_1 and WM_1 coincide in the sense that $d_{SM_1} = d_{WM_1} := d_{M_1}$ (see [3, Chapter 12, (3.7) and (3.8)]); they do not coincide on $\mathcal{D}([0,T];\mathbb{R}^k)$ for $k > 1$. By [3, Example 12.3.2] WM_1 is not metrizable on $\mathcal{D}([0,T];\mathbb{R}^k)$ for $k > 1$ but this topology is equivalent to the product topology induced by the metric d_p, with

$$d_p(x,y) := \max_{1\leq i\leq k} d_{M_1}(x_1^i, x_2^i) \text{ for } x, y \in \mathcal{D}([0,T];\mathbb{R}^k),$$

The oscillation for WM_1 is related to

$$\|z_{t_2} - [[z_{t_1}, z_{t_3}]]\| := \max_{1\leq i\leq k} |z_{t_2}^i - [z_{t_1}^i, z_{t_3}^i]|,$$

where we keep using $|\cdot|$ for the distance in one dimension while use $\|\cdot\|$ for the distance in higher dimension. Obviously, when z is monotone, $\|z_{t_2} - [[z_{t_1}, z_{t_3}]]\| = 0$. In the multi-dimensional case by monotonicity of z we mean $z_s^i \leq z_t^i$ for all $i = 1, 2, \cdots, k$ whenever $s \leq t$, or $z_s^i \geq z_t^i$ for all $i = 1, 2, \cdots, k$ whenever $s \leq t$. For details on WM_1, refer to [3, Chapter 12].

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Thus, the proofs in [1, Section 3 and Section 4.1] hold for the multi-dimensional case as long as the metric and oscillation function for SM_1 are replaced by d_p and the oscillation function for WM_1 whenever they are used. The only part deserves more discussion is [1, Section 4.2], where we used the fact that convergence in SM_1 is equivalent to the existence of convergent parameter representations; see [1, Proposition 4.7]. This equivalence is not true for WM_1. However, by the specific structure of the approximant, we can keep using SM_1 in this part. [2, Section 6] introduces sequences of stopping times and the time-changed trajectories that are exactly the parameter representations we need. By a careful inspection of [2, Section 6], [2, Lemma 6.1 and Lemma 6.2] still hold for $\mathbb{R}^k$ valued càdlàg path z. In particular, [2, Lemma 6.1 and Lemma 6.2] yield

$$z^{[n]}_{\cdot} := n \int_{\cdot-1/n}^{\cdot} z_s \, ds \to z_{\cdot} \text{ in } (\mathcal{D}((-\infty, \infty); \mathbb{R}^k), SM_1)$$

in the sense of [3, Section 12.9]. This shows that [1, Proposition 4.7] still holds with $Z_{\cdot}$ and its approximant $Z^{[n]}_{\cdot} := n \int_{\cdot-1/n}^{\cdot} Z_s \, ds$.

Note that it is sufficient to consider $c = 1$ otherwise we can consider the convergence of the integral process $\int_0^{\cdot} c_s \, dZ_s$ as a whole.

References

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