Trading under market impact
-crossing networks interacting with dealer markets-

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Abstract

We use a model with agency frictions to analyze the structure of a *dealer market* that faces competition from a *crossing network*. Traders are privately informed about their *types* (e.g. their portfolios), which is something the dealer must take into account when engaging his counterparties. Instead of participating in the dealer market, the traders may take their business to a crossing network. The dealer must take into consideration that traders have this alternative when choosing a pricing schedule. We show that the presence of a crossing network may benefit traders even if they do not trade in it. Furthermore, it results in more traders being serviced by the dealer and the book's *spread* shrinking (under certain conditions). We allow for the pricing on the dealer market to determine the structure of the crossing network, which itself influences the structure of the dealer market. This results in a feedback look that, under the same conditions that lead to a reduction of the spread, yields an equilibrium book/crossing network pair.

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Introduction

In traditional dealer markets (DM for short), liquidating large portfolios may lead to an unfavorable price impact. In response to this problem, alternative trading venues such as crossing networks (CNs for short), in which no price generation takes place but trading is opaque, have been established.[1] These two types of venues compete for liquidity, which leads to the question of how the prices and traded volumes in DMs are affected by the emergence of CNs.[2] Moreover, prices in a CN are a function of the prices in the competing DM, which in turn depend on the volume traded in the CN. In this paper we analyze the interplay between DMs and CNs within

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[1] For instance, Liquidnet and POSIT, among others.

[2] The European MiFID 2, approved in 2014 and to be implemented by January 3, 2018, introduced the new category "organized trading facility (OTF)". Among others, the OTF regime captures broker CNs. Given the increased regulatory scrutiny to which CNs are subject, analyzing their effect on primary markets is a current, relevant issue.

a principal-agent framework with private information. The principal represents a monopolistic DM run by a profit-maximizing dealer. The agents correspond to traders who choose between engaging the dealer, trading the CN or abstain from trading all together. They trade for liquidity reasons, are privately informed about their inventories and have (possibly idiosyncratic) beliefs about the probabilities of trade execution in the CN. The price in the CN depends in a pre-specified manner on the price schedule offered by the dealer and, simultaneously, it determines the traders' outside options. Within our model the CN leads to a positive externality on agents trading in the DM: independently of the distribution of types, a trader may benefit from the CN even if he takes his business to the DM. For the benchmark case of uniformly distributed types, we find that if the CN benefits only agents with larger inventories, then, in equilibrium, its introduction shrinks the spread in the DM. These effects can already be inferred from the linear adverse selection model of Mussa and Rosen (1978), where both the spread and the lowest type serviced are linearly increasing functions of the highest agent type (the one with the highest inventory in our setting). When high-type traders take their business to the CN, the highest type transacting with the dealer decreases. As a result, the spread shrinks and the dealer increases the number of low-type traders he services. In particular, more types earn positive rents. On the other hand, we show via a simple example that if the CN benefits "small traders" then the spread may widen in equilibrium.[3] This effect may be avoided if access to the CN is costly.

There are basically two branches of the literature that deal with issues related to optimal simultaneous trading in DMs and CNs by means of almost orthogonal approaches. On the one hand, starting with the contributions of Almgren and Chriss (2001) and Obizhaeva and Wang (2013), the mathematical-finance literature has extensively analyzed models of optimal trading under market impact in recent years. In this literature, it is typically assumed that trading is liquidity-driven and that it takes place under an exogenous, non-linear pricing schedule in the DM. Horst and Naujokat (2014) and Kratz and Schöneborn (2015) were the first to allow orders to be simultaneously submitted to a DM and a CN. In their models, arbitrary amounts can be submitted to CNs; order executions in CNs are exogenous. The assumption of exogenous order execution in a CN seems reasonable when trading is liquidity-driven. However, it is undesirable that there is no feedback from CN trading to the price setting in the DM. We extend the mathematical-finance literature on simultaneous trading in DMs and CNs by allowing for an impact of off-exchange trading on the prices in an associated monopolistic DM and vice versa. Since our focus is the nonlinear pricing (i.e. market impact) in the DM and not the matching mechanism in the competing CN, we do not model the latter explicitly but assume instead that the traders act on their (possibly private) beliefs about the probability of order execution.

On the other hand, the financial-economics literature is rife with equilibrium models analyzing the impact of alternative trading venues on DMs and trading behavior. For example, Parlour and Seppi (2003), study competition for order flow between heterogeneous exchanges and estab-

[3] Buti et al. (2011) provide empirical evidence that high CN activity is indeed associated with narrower spreads. Whether or not CNs lead to narrower spreads and/or welfare improvements is still subject to a controversial discussion, though.

lish that different trading architectures, such as pure limit-order and hybrid specialist/limit-order markets, can be supported in equilibrium. Glosten (1994) studies electronic limit-order books and the characteristics that other trading venues should have to successfully compete with an electronic exchange. Degryse et al. (2009) build a dynamic model of a dark pool and analyze how various transparency requirements for dark-pool orders affect traders' behavior and welfare. Daniëls et al. (2013) investigate the allocation of order flow between a DM and a CN and show how differences in traders' liquidity preferences generate a unique equilibrium in which patient traders use the CN while impatient traders submit orders directly to the DM. Buti et al. (2016) model the competition between an open limit order book and a dark pool, and focus on the interaction between dark-pool trading and characteristics of the limit order book. To simplify the analysis of market impact, this literature typically assumes that the market participants trade only a single unit of the stock. This is the main point of divergence with our model: whereas we allow for differently-sized trades but do not endogenize matching probabilities, most if not all of this literature targets the equilibrium workings of the off-exchange venues at the cost of only considering unitary trades. In particular, pricing rules are linear. For instance, in their seminal paper, Hendershott and Mendelson (2000) use a setup where multiple dealers play a Bertrand game against each other, but are passive in equilibrium. Information asymmetry corresponds to private information about the common value of the asset, which may be short or long-lived (in the sense that it may be used sequentially) and not all traders have. All traders may first submit unitary orders to the CN and, if unexecuted, may then move on to the DMs. The timing of the actions of uninformed traders depend on an exogenously-given impatience factor. This, together with the duration of private information, determines the number of trading counterparties in the CN; thus, the probability of execution and, ultimately, the equilibrium spread. The authors identify two counteracting effects of larger trading volumes in the CN. On the one hand, a *liquidity externality*: a higher trading volume in the CN increases liquidity, which benefits all traders. On the other hand, a *crowding effect*: low- and high-liquidity preference trades may compete against each other on the same side of the market. We do not find the aforementioned probability endogenously, as we do not model the matching mechanism in the CN explicitly. Instead, we assume each trader computes her expected utility of trading in each venue and then chooses which one to use. This takes into account the possibility of finding a match in the CN, as well as the price at which trades are executed there. The said expected utilities contemplate trading multiple units of the stock according to possibly non-linear pricing rules.

The remainder of this paper is organized as follows: First, in the spirit of Biais et al. (2000), we formulate the dealer's optimization problem for given execution prices in the CN. We assume each trader computes her expected utility of trading in the different venues and then chooses which one to use. This takes into account the possibility of finding a match in the CN, as well as the price at which unitary units are traded there. The dealer's objective is then to devise a pricing schedule so as to maximize his expected profits from trading, roughly defined as the gains from trading certain positions net of the associated costs. We show that this problem has a unique solution on the set of traders participating in the DM.

Second, we study the qualitative influence of a CN on an existing market, as introducing a CN adds constraints to the dealer's optimization problem. We gauge the impact of these constraints using the Lagrangian techniques introduced in Jullien (2003). This allows us to identify traders who defect the DM and those who previously did not trade in it but, by virtue of more appealing conditions, now engage the dealer. As a consequence, we fully describe the DM via a non-linear pricing rule and determine the volume of trading in each venue. We prove that, under certain conditions, the presence of the CN results in a positive externality on agents trading in the DM: more traders earn positive rents and the spread in the DM shrinks. We illustrate these results by means of several examples with and without a CN.

Finally, having understood the dealer's problem with exogenous CN prices, we proceed to show the existence of equilibrium prices. The first step is to specify a *price generation mechanism* via which prices in the DM determine those in the CN. Given an exogenous sell/buy price pair (π_-, π_+) in the CN, the dealer optimally chooses his pricing schedule, which will most likely not induce (π_-, π_+) through the price generation mechanism. We identify conditions under which this iterative procedure converges to a fixed point: an equilibrium price schedule is such that the optimal reaction of the dealer to the prices (π_-, π_+) in the CN induces again (π_-, π_+).[4] Our study of such a feedback loop is novel and it is a crucial component in our analysis of the interactions between DMs and CNs, which is typically not unidirectional. As an application we consider a problem of optimal portfolio liquidation where traders can chose between a DM and a CN (in this particular case a dark pool). We show the presence of the DP leads to a shrinkage of the spread and prove the existence of an equilibrium price .

Summarizing, our main results are: *i*) identifying sufficient conditions for the spread in the DM to shrink in the presence of a CN, and *ii*) identifying sufficient conditions for the existence of equilibrium prices. We conclude this introduction with a statement from Section 7.3 in Gomber et al. (2017), which coincides with our findings:

> "It is also possible that all types of dark pool trading activity may not have a uniform impact on the markets, given the different types of market structure that are clubbed in its definition."

1. The model

We consider a quote-driven market for an asset, in which a risk-neutral *dealer* engages a group of privately-informed *traders*. The dealer market (DM for short) is described by a pricing schedule $T : \mathbb{R} \to \mathbb{R}$, where q units of the asset are offered to be traded for the amount $T(q)$. As long as $q = 0$ is traded in the DM, we may follow the standard approach in Biais et al. (2000) and

[4] Put differently, this work is based on a particular setting, in which the competitor to the monopoly is not modeled, and instead adopts an automatic quotation system inspired by some existing financial markets.

assume that $T(0) = 0$ and that there is an increasing function $t(\cdot)$ such that

$$T(q) = \int_0^q t(s)ds, \quad q \geq 0, \tag{1}$$

and analogously for $q \leq 0$. In other words, the marginal price at which the s-th unit is traded is $t(s)$, which justifies the monotonicity assumption on $t(\cdot)$. However, there may be situations where the smallest positive quantity traded in the DM, say $\overline{q}_0$, is strictly positive.[5] In that case there exists $\overline{T} > 0$ such that, for $q \geq \overline{q}_0$,

$$T(q) = \overline{T} + \int_{\overline{q}_0}^q t(s)ds,$$

and analogously for $q \leq \underline{q}_0$. $\overline{T}$ cannot be decomposed as an integral because we have no precise information about non-traded quantities. In general, pricing schedules are not differentiable at $\underline{q}_0$ and $\overline{q}_0$. We have that the *spread* is

$$\mathcal{S}(T) := \left|T'(\overline{q}_0+) - T'(\underline{q}_0-)\right| = \left|t(\underline{q}_0+) - t(\underline{q}_0-)\right|,$$

where $t(\underline{q}_0-)$ and $t(\underline{q}_0+)$ are the *best-bid* and *best-ask* prices for traded quantities, respectively.[6] In particular, if $\overline{q}_0 = \underline{q}_0 = 0$ then the spread is $\mathcal{S}(T) = |T'(0+) - T'(0-)| = |t(0+) - t(0-)|$, with corresponding best-bid and best-ask prices $t(0-)$ and $t(0+)$. Below we establish conditions under which small quantities are traded in the DM and mostly focus in this case.

The dealer is exposed the inventory or risk costs $C(q)$ associated with each position q and her corresponding profits from trading are $T(q) - C(q)$. We assume that C is convex and $C(0) = 0$. An archetypical example are quadratic costs: $C(q) = \alpha \cdot q^2$, with $\alpha > 0$.

The traders' idiosyncratic characteristics are indexed by $\theta \in \Theta := [\underline{\theta}, \overline{\theta}]$. Saying that a trader's *type* is θ means that if she trades q shares for $T(q)$ dollars, his utility is given by the smooth function

$$u(\theta, q) - T(q) := \theta\psi_1(q) + \psi_2(q) - T(q),$$

with $\psi_1(0) = \psi_2(0) = 0$.[7] Having both buyers and sellers requires us to assume $\underline{\theta} < 0 < \overline{\theta}$.

Besides participating in the DM, each trader may submit an order to a *crossing network* (CN for short). This is an alternative trading venue where trades take place at fixed bid/ask prices $\pi := \big(\pi_-, \pi_+\big)$, but where execution is not guaranteed. For a specific π, the quantity $\mathrm{w}(\theta; \pi)$, which may be negative, represents the expected utility of the θ-type investor who decides to trade in the CN.[8] Following Daniëls et al. (2013) and H-M we focus on the case where a trader

[5] See Assumption 1.1 and the discussion that follows it.

[6] We use the notation $\overline{q}_0+$ and $\underline{q}_0-$ to denote right and left limits, respectively.

[7] The linearity in θ of the traders' utility is necessary for the convex-analytic techniques that we use below. Although not without loss of generality, many interesting cases can be phrased in this framework (see Section 4 for and example where θ represents an inventory position in a portfolio-liquidation setting). Striving for more generality would require the use of u-convex analysis as in Carlier (2001).

[8] Given that $\mathrm{w}(\cdot; \pi)$ corresponds to an expected utility, it incorporates the probability of non-execution.

chooses **exclusively** between abstaining from trading or doing it either in the DM or the CN, i.e., we do not allow for simultaneous participation in the DM and the CN. We initially take π as given but later analyze how it is endogenously determined through a feedback loop between the DM and the CN. It is key to our analysis that the dealer is able to match the utilities that traders enjoy in the CN, even if this comes at a loss. As we show below, this requires $\mathrm{w}(\cdot;\pi)$ to be a convex function. Finally, we assume there is a fixed cost of entry $\kappa > 0$ to the CN such that "small" traders do not benefit from off-market participation.[9] More precisely, we work under the following technical assumption:

Assumption 1.1. *The traders' expected utilities from participating in the CN have the form* $\mathrm{w}(\cdot;\pi) = \widetilde{\mathrm{w}}(\cdot;\pi) - \kappa$, *where* $\widetilde{\mathrm{w}}(\cdot;\pi)$ *is a convex function that satisfies* $\widetilde{\mathrm{w}}(0;\pi) = 0$ *and* $\kappa > 0$ *is the fixed cost of accessing the CN.*

We can see the way in which the execution price π of the CN enters the traders' expected utility from trading in it. When π itself is a function of the pricing schedule T of the DM, which the dealer chooses taking $\mathrm{w}(\cdot;\pi)$ into account, we obtain the aforementioned feedback loop. We prove in Lemma B.1 that under Assumption 1.1 the pricing schedules are as in Expression 1.

Remark 1.2. *The cost of entry κ plays a crucial role in our result concerning the narrowing of the spread in the presence of a CN. In its absence, we observe that the spread may actually widen (see Example 2.7). This result is not dissimilar to Proposition 16 in H-M, where no cost of access plus short-lived information lead to a higher spread in the presence of the CN. Our analysis then suggests that having a cost of access is in fact desirable, as otherwise the CN may harm price discovery. We may conclude that CNs may or may not harm price discovery, depending on who trades in them.*

The traders' third option is to abstain from trading altogether, which can only be the case for types such that $\mathrm{w}(\theta;\pi) < 0$. Given that, in terms of utilities and costs, this is equivalent to trading the quantity $q = 0$ at zero price in the DM, we assume that abstaining types trade "nothing for nothing" with the dealer. This simplifies the modeling without having any impact on incentives. In the sequel we refer to $u_0(\cdot;\pi) := \max\{\mathrm{w}(\cdot;\pi), 0\}$ as the traders' *outside option(s).*

Trading in the DM is anonymous: the dealer is unable to determine a trader's type before he engages her. However, the dealer has ex-ante beliefs about is the distribution of types over Θ, which are described by density $f : \Theta \to \mathbb{R}_+$. Below we specify the traders' and the dealer's optimization problems and analyze the impact of the CN on the DM, especially on its spread and the rents earned by agents who trade in it.

Admittedly, we do not consider the case where this probability depends on the orders submitted by other traders.

[9] Having direct costs of trading the CN is not an uncommon assumption. For instance, the model in H-M contemplates an access cost c_0 and an execution one c_e.

1.1. The traders' problem

Until further notice we consider π to be fixed. Given the pricing schedule T, the problem of a trader of type θ is to determine,

$$q_m(\theta) := \operatorname*{argmax}_{q \in \mathbb{R}} \left\{ u(\theta, q) - T(q) \right\}$$

and then choose, for $q_m \in q_m(\theta)$, between his *indirect-utility* $v(\theta) := u\big(\theta, q_m\big) - T\big(q_m\big)$ from trading in the DM and his outside option $u_0(\theta; \pi)$. As the supremum of affine functions, the indirect utility function is convex (thus the need for u_0 to be convex if we wish to analyze a situation where the dealer can match the CN). Each pricing schedule induces a segmentation of the type space. A trader of type θ *participates* in the DM if $v(\theta) \geq u_0(\theta; \pi)$, assuming that ties are broken in the dealer's favor. Conversely, a trader of type θ *is excluded* from trading in the DM if $v(\theta) < u_0(\theta; \pi)$. For a given schedule T, we denote the set of excluded types by $\Theta_e(T; \pi)$. A trader is *fully serviced* if she earns strictly positive profits from interacting with the dealer.

1.2. The dealer's problem

There is no loss of generality in assuming that the DM is described by *books* of the form $(q, \tau) = \big\{ \big(q(\theta), \tau(\theta)\big), \theta \in \Theta \big\}$, where $\tau : \Theta \to \mathbb{R}$ is absolutely continuous.[10] We then write $\Theta_e(q, \tau; \pi)$ instead of $\Theta_e(T; \pi)$ for the set of excluded types. A trader of type θ could misrepresent his type by choosing a contract $\big(q(\widetilde{\theta}), \tau(\widetilde{\theta})\big)$ intended for traders of type $\widetilde{\theta} \neq \theta$. The dealer strives to avoid this situation because he wants to exploit the information contained in the density of types. This requires that he offers *incentive-compatible* books, i.e. those that satisfy

$$\max_{\widetilde{\theta} \in \Theta} \left\{ u\big(\theta, q(\widetilde{\theta})\big) - \tau(\widetilde{\theta}) \right\} = u\big(\theta, q(\theta)\big) - \tau(\theta),$$

which we shorthand by writing $(q, \tau) \in \mathbf{IC}$.

The dealer's objective is to maximize his expected income from engaging the traders. Taking into account the impact of the CN on the traders' optimal actions, his problem is to devise (q^*, τ^*) so as to solve

$$\mathcal{P}(\pi) := \sup_{(q,\tau) \in \mathbf{IC}} \int_{\Theta_e^c(q,\tau;\pi)} \Big(\tau(\theta) - C\big(q(\theta)\big) \Big) f(\theta) d\theta.$$

Remark 1.3. *The dealer's cons function depends on traded quantities but not on the trader's type. As a result, our model should be viewed as one where trading is liquidity-driven and where traders have no private information on the value of the stock.*

From the *Envelope Theorem*, if a book $\big\{ \big(q(\theta), \tau(\theta)\big), \theta \in \Theta \big\}$ is incentive compatible, then $q(\theta) = \psi_1^{-1}\big(v'(\theta)\big)$ for almost all $\theta \in \Theta$. Therefore, starting from a convex indirect-utility

[10] The *Revelation Principle* (see, e.g. Myerson (1991)) states that, when studying outcomes in hidden-information, monoplolistic games such as ours, there is no loss of generality in focusing on direct-revelation mechanisms, i.e. those mechanisms where the set of types indexes the books. Furthermore, from the *Taxation Principle* (see e.g. Rochet (1985)) there is also no loss of generality in writing $\tau(\theta)$ instead of $T(q(\theta))$.

function we can recover the quantities in the incentive-compatible book that generated. This allows us to write

$$v(\theta) = \theta\, v'(\theta) + \psi_2 \circ \psi_1^{-1}\big(v'(\theta)\big) - \tau(\theta). \tag{2}$$

We see that the traders' indirect utility function contains all the information about the quantities and the pricing schedule and $\Theta_e^c(q,\tau;\pi) = \Theta_e^c(v;\pi)$. Introducing the functions

$$\widetilde{K}(q) := C\big(\psi_1^{-1}(q)\big) - \psi_2 \circ \psi_1^{-1}(q) \quad \text{and} \quad i(\theta, v, q) := \theta \cdot q - v - \widetilde{K}(q)$$

and denoting by $\mathcal{C}$ the set of all real-valued convex functions over Θ, we can restate the dealer's problem as

$$\mathcal{P}(\pi) = \sup_{v \in \mathcal{C}} \int_{\Theta_e^c(v;\pi)} i\big(\theta, v(\theta), v'(\theta)\big) f(\theta) d\theta.$$

We prove in Appendix A that, under suitable assumptions, Problem $\mathcal{P}(\pi)$ admits a solution that is unique on the set of participating types (we say that it is quasi-unique). Even though there is no uniqueness on the set of excluded types, from the traders' point of view there is no ambiguity: they either trade in the DM or they take their outside option. The non-uniqueness is also a non-issue for the dealer because it only pertains types that he does service. In the sequel we denote by $v(\cdot;\pi)$ "the" solution to Problem $\mathcal{P}(\pi)$. For any $v \in \mathcal{C}$, we refer to

$$\Theta_0(\pi) := \big\{\theta \in \Theta \,|\, v(\theta;\pi) = 0\big\}$$

as the set of *reserved traders.* By convexity $\Theta_0(\pi)$ coincides with some interval $[\underline{\theta}_0(\pi), \overline{\theta}_0(\pi)]$ and it is precisely at its endpoints where $t(0-)$ and $t(0+)$ are determined.

Remark 1.4. *There are several instances where the proofs of our results concern conditions on points to the left of $\underline{\theta}_0(\pi)$ or to the right of $\overline{\theta}_0(\pi)$ that are analogous. So as to streamline the presentation, whenever we find ourselves in one of these "either-or" situations, we deal only with the positive case. Below we compare scenarios with and without the presence of a CN and use the subindexes "m" and "o" to distinguish between structures or quantities with and without a CN, respectively.*

2. The impact of a crossing network

In this section we look at the impact that a CN has on the spread, on participation and on the traders' welfare.

2.1. A benchmark without a CN

We first analyze the benchmark case where the traders do not have access to a CN. The corresponding dealer's problem is denoted by $\mathcal{P}_o$. The dealer's income for a given indirect utility function $v \geq 0$ is

$$I[v] := \int_{\Theta} i\big(\theta, v(\theta), v'(\theta)\big) f(\theta) d\theta.$$

The Lagrangian for the dealer's problem incorporates the non-negativity constraint:

$$\mathcal{L}(v,\gamma) := I[v] + \underbrace{\int_\Theta v(\theta)d\gamma(\theta)}_{:=\langle v,\gamma\rangle}, \quad v \in \mathcal{C}.$$

The complementary-slackness conditions $\langle v,\gamma\rangle = 0$ and $d\gamma(\theta) = 0 \Rightarrow v(\theta) > 0$ tell us that identifying the types for which $d\gamma(\theta) > 0$ is equivalent to determining those traders who do not engage the dealer. In the upcoming sections, the analysis of the market's segmentation and the spread ensues by determining the optimal γ, which can be assumed to satisfy $\gamma(\overline{\theta}) = 1$.[11] Integrating by parts, we may rewrite $\mathcal{L}(v,\gamma)$ as

$$\mathcal{L}(q,\gamma) = \int_\Theta \Big(\Big(\theta + \frac{F(\theta)-\gamma(\theta)}{f(\theta)}\Big)\psi_1\big(q(\theta)\big) - \widetilde{C}\big(q(\theta)\big)\Big) f(\theta)d\theta,$$

where $q(\theta) = \psi_1^{-1}\big(v'(\theta)\big)$ and $\widetilde{C}(q) := C(q) - \psi_2(q)$. The first step in determining the optimal γ is to maximize

$$q \mapsto \sigma(\theta,q,\Gamma) := \Big(\theta + \frac{F(\theta)-\Gamma}{f(\theta)}\Big)\psi_1(q) - \widetilde{C}(q)$$

pointwise for a fixed Γ.[12] The sought-after maximizer is[13]

$$l(\theta,\Gamma) := K^{-1}\Big(\frac{F(\theta)+\theta f(\theta)-\Gamma}{f(\theta)}\Big), \quad \text{where } K(q) := \widetilde{C}'(q)/\psi_1'(q).$$

For each $\theta \in \Theta$ and $\Gamma \in [0,1]$, the quantity $l(\theta,\Gamma)$ is a candidate for the optimal $q(\theta)$ and incentive compatibility is verified if the mapping $\theta \mapsto l(\theta,\Gamma)$ is increasing. This showcases the connection between q and γ.

Regularity properties of the solutions to variational problems subject to convexity constraints were studied in Carlier and Lachand-Robert (2001). Their methodology can be directly adapted to prove the following result, which formalizes the *vox populi* saying "quality does not jump".

Proposition 2.1. *If $v \in \mathcal{C}$ is a stationary point of $\mathcal{L}(v,\gamma)$, then it is continuously differentiable.*

The fact that, at the optimum, the mapping $\theta \mapsto v'(\theta)$ is continuous, implies that q is also a continuous function of the types. From Lemma A.3 we have that the quantity $q(\underline{\theta}) < 0$. Given that quality does not jump, the complementary-slackness condition implies there is some $\widetilde{\theta} > \underline{\theta}$ such that $\gamma(\theta) = 0$ for $\theta \in [\underline{\theta}, \widetilde{\theta})$. We determine $\underline{\theta}_0$ by identifying the type where γ starts to grow. This requires solving the equation

$$K^{-1}\Big(\theta + \frac{F(\theta)}{f(\theta)}\Big) = 0.$$

[11]In technical terms the Lagrange multiplier γ belongs to the space of non-negative functions of bounded variation $BV_+(\Theta)$. It follows from Pontryagin's Maximum Principle and the fact that f is a probability density function that there is no loss of generality in assuming that $\gamma(\overline{\theta}) = 1$.

[12]We use the notation Γ whenever dealing with an arbitrary but fixed value of γ.

[13]See Appendix A, in particular Assumption A.1, for conditions that guarantee the uniqueness of $l(\theta,\Gamma)$.

Furthermore, as v must be convex, once $v(\hat{\theta}) > 0$ then $v(\theta) > 0$ for all $\theta > \hat{\theta}$. Hence, $\overline{\theta}_0$ is determined by solving the equation

$$K^{-1}\Big(\theta - \frac{1-F(\theta)}{f(\theta)}\Big) = 0.$$

Sufficient conditions for the mapping $\theta \mapsto l(\theta, \Gamma)$ to be non-decreasing are that the *Hazard rates* satisfy

$$\frac{d}{d\theta}\left(\frac{F(\theta)}{f(\theta)}\right) \geq 0 \geq \frac{d}{d\theta}\Big(\frac{1-F(\theta)}{f(\theta)}\Big), \tag{3}$$

see Biais et al. (2000) for a discussion on this condition. Let us assume that we have determined Θ_0. What remains is then to connect the participation constraint with the spread. Differentiating Eq. (2) and noting that $v'(\theta) = \psi_1(q(\theta))$ we have that

$$\tau'(\theta) = q'(\theta)\big(\theta\psi_1'(q(\theta)) + \psi_2'(q(\theta))\big).$$

Observe that $\tau'(\underline{\theta}_0)$ and $\tau'(\overline{\theta}_0)$ are in fact $T'(0-)$ and $T'(0+)$ as, by construction, $q(\underline{\theta}_0) = q(\overline{\theta}_0) = 0$. If we define $\phi_1 := \psi_1'(0)$ and $\phi_2 := \psi_2'(0)$, then we have that the spread is determined by the expressions

$$t(0-) = q'(\underline{\theta}_0-)\big(\underline{\theta}_0\phi_1 + \phi_2\big) \quad \text{and} \quad t(0+) = q'(\overline{\theta}_0+)\big(\overline{\theta}_0\phi_1 + \phi_2\big). \tag{4}$$

Our objective in Section 2.2 is to compare these values to those obtained in the presence of a crossing network. For simplicity we define $\Theta_o := \Theta_0(v_o)$, where v_o solves problem $\mathcal{P}_o$.

Before we proceed we present two examples that illustrate the use of the methodology described hitherto. The first revisits Mussa and Rosen (1978). The second is slightly more advanced. We use it below to illustrate the complex structure of optimal pricing schedules and utilities in the presence of CNs.

Example 2.2. *Let us assume that $\Theta = [-r, r]$ for some $r > 0$, that types are uniformly distributed and that*

$$u(\theta, q) = \theta q.$$

We also set $C(q) = 0.5\,q^2$. Given that a trader of type $\theta \in \Theta_o$ is brought down to reservation utility and hence trades $q(\theta) = 0$, the expression

$$q(\theta) = \theta + \frac{F(\theta) - \gamma(\theta)}{f(\theta)} = 2\theta + r - 2r\gamma(\theta)$$

implies that the Lagrange multiplier is

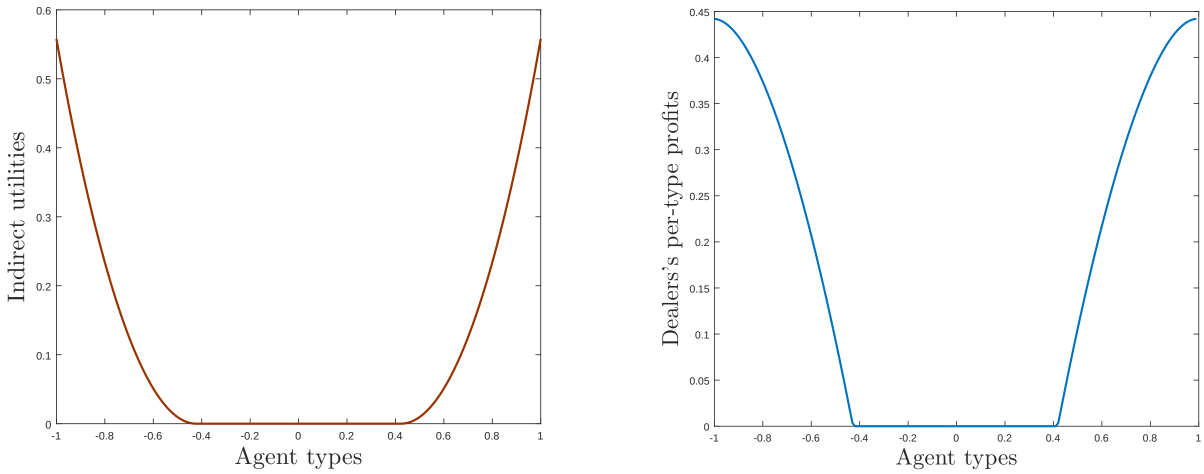
$$\gamma(\theta) = \begin{cases} 0 & ,\theta < \underline{\theta}_0; \\ \frac{1}{2} + \frac{\theta}{r} & ,\theta \in \Theta_o; \\ 1 & ,\theta > \overline{\theta}_0. \end{cases}$$

By direct computation we find that $\underline{\theta}_0 = -\frac{r}{2}$ *and* $\overline{\theta}_0 = \frac{r}{2}$*. In particular,* $q'(\underline{\theta}_0-) = q'(\overline{\theta}_0+) = 2$ *and hence* $t(0-) = -r$ *and* $t(0+) = r$*. Thus, the spread increases linearly in the highest/lowest type.*[14]

Example 2.3. *Let us assume that the distribution of types over* $\Theta = [-1, 1]$ *is given by* $f(\theta) = (2\theta + 3)/4$ *for* $\theta \in [-1, 0)$ *and* $f(\theta) = (3 - 2\theta)/4$ *for* $\theta \in [0, 1]$*; that* $C(q) = 0.5\, q^2$ *and that* $u(\theta, q) - \tau = \theta \cdot q + 0.25\, q^2 - \tau$*. It is straightforward to show that the conditions on the Hazard rates are satisfied and that*

$$K^{-1}\Big(\theta + \frac{F(\theta)}{f(\theta)}\Big) = 2\Big[\frac{3\theta^2 + 6\theta + 2}{2\theta + 3}\Big] \quad \textit{and} \quad K^{-1}\Big(\theta - \frac{1 - F(\theta)}{f(\theta)}\Big) = 2\Big[\frac{3\theta^2 - 6\theta + 2}{2\theta - 3}\Big].$$

Furthermore, $\Theta_o \approx \big[-0.423, 0.423\big]$*. For the spread, we have that* $t(0-) = q'(\underline{\theta}_0)\underline{\theta}_0 \approx -1.359$ *and* $t(0+) = q'(\overline{\theta}_0)\overline{\theta}_0 \approx 1.359$*. In order to obtain* v *we integrate* q $(\psi_1(q) = q)$ *and take into account that* $v \equiv 0$ *over* Θ_o*. We plot* $\text{graph}\{v_o\}$ *in Figure 1, as well as the per-type profits of the dealer.*



(a) Indirect Utilities (b) Profits

Figure 1: An example without a crossing network

2.2. Introducing a crossing network

We now analyze the dealer's problem when there is a CN that gives a trader of type θ the expected utility $u_0(\theta; \pi)$. Recall that the dealer's problem is

$$\mathcal{P}(\pi) = \sup_{v \in \mathcal{C}} \int_{\Theta} \Big(\theta\, v'(t) - v(t) - \widetilde{K}\big(v'(\theta)\big)\Big) \mathbb{1}_{\{\Theta_e^c(v)\}}(\theta) f(\theta) d\theta.$$

Dealing with the presence of the zero-one indicator function $\mathbb{1}_{\{\Theta_e^c\}}$ is quite cumbersome (see, e.g. Horst and Moreno-Bromberg (2011)) because its domain of definition may change with

[14] This suggests that spreads decrease if high- and low-type agents opt for an outside option if present; thus, in equilibrium they do not engage the dealer.

different book choices. In contrast to the setting studied in Horst and Moreno-Bromberg (2011), however, here the CN is passive. This lack of non-cooperative-games component allows for an alternative way to proceed, which, as mentioned in Section 1, has as a key requirement that, disregarding negative expected unwinding costs, the dealer is able to match the CN. As a consequence of Assumption 1.1 and the structure of u we can show this is always possible. More specifically

Proposition 2.4. *There exists an incentive compatible book* $\big(q_c, \tau_c\big)$ *such that for almost all* $\theta \in \Theta$ *it holds that* $u\big(\theta, q_c(\theta)\big) - \tau_c(\theta) = u_0(\theta; \pi)$.

Observe that finding the incentive compatible book $\big(q_c, \tau_c\big)$ that replicates $u_0(\cdot; \pi)$ does not tell us anything about $\tau_c(\theta) - C(q_c(\theta))$, which may be negative. In other words, matching the CN for all types may result in type-wise losses. With Proposition 2.4 in hand, we may make use of the following *accounting trick*, which was introduced in Jullien (2003): let us assume that the dealer had access to a fictitious market such that the unwinding costs from trading in it, denoted in the sequel by C_c, satisfy $C_c(q(\theta)) = \tau(\theta)$. In this way, we may again assume that the dealer trades with all market participants, but now his costs of unwinding are given by the function $\mathbb{C} : \mathbb{R} \to \mathbb{R}$ defined as

$$\mathbb{C}(q) := \min\big\{C(q), C_c(q)\big\}, \quad q \in \mathbb{R}.$$

In terms of incentives, nothing is distorted by introducing the cost function $\mathbb{C}$, but we must identify the points where there is switching from using C to using C_c and vice versa. These switching points determine the market's segmentation.

If we define, for any traded quantity q, the function $\widetilde{\mathbb{C}}(q) := \mathbb{C}\big(q\big) - \psi_2\big(q\big)$, then we may re-use the machinery from Section 2.1 with minor modifications;[15] namely, denoting by $\mathbb{I}$ the dealer's utility corresponding to the cost function $\mathbb{C}$, we may write the Lagrangian of the dealer's problem as

$$\mathbb{L}(v, \gamma) := \mathbb{I}[v] + \langle v - u_0(\cdot; \pi), \gamma\rangle,$$

with corresponding complementary-slackness conditions $\langle v - u_0(\cdot; \pi), \gamma\rangle = 0$ and $d\gamma(\theta) = 0 \Rightarrow v(\theta) > u_0(\cdot; \pi)$. As before, identifying the types for which $d\gamma(\theta) > 0$ is equivalent to determining those traders who take their outside option.[16] Whether these types are reserved or excluded, however, depends on the strict positivity of $u_0(\cdot; \pi)$. From here on, we may proceed as in Section 2.1 in order to find the quantities that the dealer chooses to offer. Strictly speaking we should find the pointwise maximizer in q of the expression

$$\Big(\theta + \frac{F(\theta) - \Gamma}{f(\theta)}\Big)\psi_1(q) - \mathbb{K}(q),$$

where $\mathbb{K}(q) := \widetilde{\mathbb{C}}(q) - \psi_2(q)$. This may fortunately be avoided, given that whenever $\mathbb{C}(q) = C_c(q)$,

[15] Observe that Assumption 1.1 and Proposition 2.4 imply that $\widetilde{\mathbb{C}}$ satisfies Assumption A.1.
[16] See Theorem 2 in Jullien (2003) and the discussion that follows.

the participation constraint binds and $q(\theta) = q_c(\theta)$. Next, we study the connection between the solution to the fictitious problem $\mathbb{P}(\pi)$ and that to $\mathcal{P}(\pi)$.

Whenever the participation constraint does not bind, the dealer selects the quantity to be chosen via the pointwise maximization of the mapping $q \mapsto \sigma(\theta, q, \Gamma)$. What makes the current problem trickier than the case without a CN is that now we must pay more attention to the evolution of the multiplier γ. If we compare $l(\theta, 0)$ and $l(\theta, 1)$ to $q_c(\theta)$ we may pinpoint the set where the participation constraint may bind. Observe that $\{l(\theta, 1), \theta \in \Theta\}$ and $\{l(\theta, 0), \theta \in \Theta\}$ are the sets of the lowest and highest quantities the dealer may offer in an individually-rational way. Hence, as long as $l(\theta, 1) \leq q_c(\theta) \leq l(\theta, 0)$, there is the possibility of *profitable matching.*

There might be instances where the participation constraint is binding for some type $\theta \in \Theta$, i.e. $\big(q(\theta), \tau(\theta)\big) = \big(q_c(\theta), \tau_c(\theta)\big)$, and $\tau_c(\theta) - C\big(q_c(\theta)\big) < 0$. In such cases $\mathbb{C}\big(q_c(\theta)\big) = C_c\big(q_c(\theta)\big)$ and $\theta \in \Theta_e(v)$ for the corresponding indirect utility function, and we say there is *exclusion.*

Remark 2.5. *It is at this point that the quasi-uniqueness mentioned in Section 1.2 can be addressed. The dealer's problem $\mathbb{P}(\pi)$ using the cost function $\mathbb{C}$ results in the condition*

$$\big(i(\theta, v(\theta), v'(\theta))\big)_+ = \big(i(\theta, v(\theta), v'(\theta))\big)$$

being trivially satisfied. As a consequence, problem $\mathbb{P}(\pi)$ admits a unique solution. The latter coincides, by construction, with the solution to $\mathcal{P}(\pi)$ whenever $\mathbb{C}(q(\theta)) = C(q(\theta))$. The caveat is that the solution to problem $\mathbb{P}(\pi)$ is blind towards what is offered to excluded types, as their outside option is costlessly matched (they are effectively reserved). Constructing incentive compatible contracts for the excluded types is, thanks to the convexity of the indirect utility function, relatively simple. For instance if an interval of types (θ_1, θ_2) were excluded (but θ_1 and θ_2 participated) one could consider any two supporting lines to graph$\{v(\cdot; \pi)\}$ at $(\theta_1, v(\theta_1; \pi))$ and $(\theta_2, v(\theta_2; \pi))$. From the resulting indirect-utility function on (θ_1, θ_2) one could extract the corresponding quantities and prices. The resulting global convexity of the indirect-utility function offered by the dealer would imply that all incentives would remain unchanged. Whether the dealer would suffer losses from the contracts offered to types on (θ_1, θ_2) would be irrelevant, given the corresponding traders would not participate.

As mentioned above, here it is not necessary to determine $\gamma(\theta)$ in order to do likewise with $q(\theta)$. On the other hand, however, if we interpret γ as the shadow cost of satisfying the participation constraint, we may wish to identify the multiplier so as to have a measure of the impact of the CN on the dealer's profits. The following result, which deals with points where there is switching between matching and fully servicing, extends Proposition 2.1.

Proposition 2.6. *For $\pi \in \mathbb{R}^2$ given, let $\widetilde{\theta} \in \Theta$ be such that there exists $\epsilon > 0$ such that $v(\theta; \pi) = u_0(\theta; \pi)$ on $(\widetilde{\theta} - \epsilon, \widetilde{\theta}]$ and $v(\theta; \pi) > u_0(\theta; \pi)$ on $(\widetilde{\theta}, \widetilde{\theta} + \epsilon]$. Furthermore, assume that*

$$\int_{\widetilde{\theta}-\epsilon}^{\widetilde{\theta}} \big(\tau(\theta) - C(q(\theta))\big) f(\theta) d\theta > 0,$$

where $\{(q(\theta), \tau(\theta)), \theta \in \Theta\}$ *implements* $v(\cdot;\pi)$. *In other words, there is profitable matching on* $(\widetilde{\theta}-\epsilon, \widetilde{\theta}]$ *and the dealer fully services types on* $(\widetilde{\theta}, \widetilde{\theta}+\epsilon]$. *Then* $\partial v(\widetilde{\theta};\pi)$ *is a singleton. The result also holds if the order of the matching and full-servicing intervals is switched.*

The rationale behind Proposition 2.6 is that, as long as the dealer is able to match the traders' outside option without incurring in a loss, it is possible to normalize the latter to zero and directly apply Proposition 2.1. This is, naturally, not the case when matching u_0 results in losses. We put Proposition 2.6 to work in Example 2.12. We show in the following example how, in the absence of Assumption 1.1, the presence of the CN may lead to a widening of the spread.

Example 2.7. *We revisit Example 2.2 with* $r = 1$, *i.e. the traders' types span* $[-1, 1]$ *and they are uniformly distributed. We consider two different expected utilities from trading in the CN; namely,*

$$\widetilde{\mathrm{w}}\big(\theta; (-0.05, 0.05)\big) = \begin{cases} -0.05\,\theta, & \text{if } \theta \le 0; \\ 0.05\,\theta, & \text{if } \theta > 0; \end{cases}$$

and $\mathrm{w}\big(\theta, (-0.05, 0.05)\big) = \widetilde{\mathrm{w}}\big(\theta, (-0.05, 0.05)\big) - 1/50$. *In other words, these two expected utilities from trading in the CN differ only in the access cost (absent in* $\widetilde{\mathrm{w}}$*). Clearly, the dealer can match* $\widetilde{\mathrm{w}}$ *by offering the incentive-compatible book* $\big((-0.05, 0), (0.05, 0)\big)$. *However, trading either of this contract with any of the dealers will result in a loss. As a consequence, there are no reserved traders and the type space is partitioned only into a set of excluded types (in this case* $(\underline{\theta}_e, \overline{\theta}_e) = (-0.55, 0.55)$*) and one of fully serviced ones (in this case* $[-1, \underline{\theta}_e] \cup [\overline{\theta}_e, 1]$*). The first positive quantity traded in the DM is* $q = 0.1003$ *for* $T(0.1003) = 0.0277$. *Therefore the pricing schedule in the DM is, for* $q \geq 0.1003$,

$$T(q) = 0.0277 + \int_{0.1003}^{q} t(s)ds,$$

where $t(s) = 2(v')^{-1}(s)$ *and* v *is the corresponding indirect utility function. In particular, the spread is given by (the situation for* $\theta < 0$ *is symmetric as in Example 2.2)*

$$\mathcal{T} = \big|t(0.1003) - t(-0.1003)\big| = 2\big|0.55 - (-0.55)\big| = 2.2;$$

whereas the spread in Example 2.2 for $r = 1$ *equals two. In other words, we observe a widening of the spread in the presence of a CN. This is due to the absence of access costs. If we repeat the previous exercise, but now using* $\widetilde{w}$ *as the expected utility of trading in the CN, we have that the set of reserved types is* $\Theta_0 = (-0.4125, 0.4125)$, *no types are excluded and the spread is* 1.625. *We present the ask side of the DM in Figure 2.*

Before moving on, we present below a modification to Example 2.3 that shows how even traders without access to a non-trivial outside option benefit from the presence of the CN and that the optimal Lagrange multiplier need not be continuous.

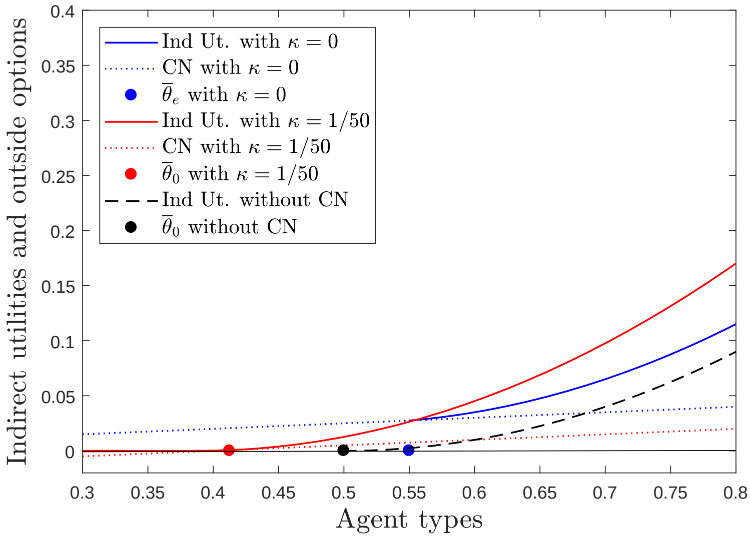


Figure 2: The indirect-utility functions and CN expected utilities with(out) access costs.

Example 2.8. *Let f, Θ, C and u be as in Example 2.3 and assume that the CN offers the traders the following expected profits:*

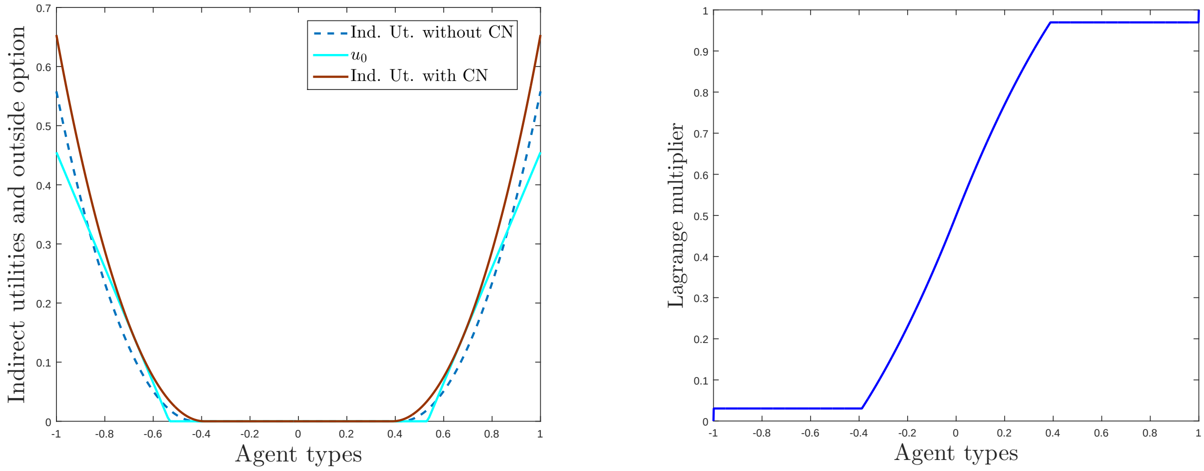
$$\mathrm{w}(\theta; (3.2, 3.2)) = \begin{cases} -0.975\theta - 0.52, & \text{if } \theta \leq -\frac{8}{15}; \\ 0.975\theta - 0.52, & \text{if } \theta \geq \frac{8}{15}; \\ \text{convex and negative for } \theta \in (-\frac{8}{15}, \frac{8}{15}). \end{cases}$$

Matching this outside option would require the dealer to offer the book $(\pm 0.975, 0.52)$. This is profitable, hence the indirect utility never lies below u_0. To illustrate this, we have plotted the indirect-utility function in Figure 3(a). It strictly dominates the one plotted in Figure 1(a) for all types who earn positive profits. The smooth pasting condition ($l(\theta, \gamma(\theta)) = q_c(\theta)$ where v touches u_0, i.e. in ± 0.675) determines the optimal Lagrange multiplier, namely $\gamma(-1) = 0$ and $\gamma \equiv 0.030$ on $(-1, -0.389]$. For positive types we obtain symmetrically $\gamma(1) = 1$ and $\gamma \equiv 0.970$ on $[0.389, 1)$. The new spread, given by $\big(t(0-), t(0+)\big) = (-1.282, 1.282)$, is strictly smaller than in the case without a CN.

The following theorem, one of our main results, analyzes the impact of the CN on the DM and the traders' welfare. Its proof can be found in AppendixB.

Theorem 2.9. *For a given price $\pi = (\pi_-, \pi_+)$ let $\mathcal{S}_m$ and $\mathcal{S}_o$ be the spreads with and without the presence of the crossing network and v_o and $v(\cdot; \pi)$ the corresponding indirect-utility functions, respectively. In the presence of the crossing network*

1. *less types are reserved, i.e. $\Theta_0(v_o) \supseteq \Theta_0(\pi)$. Furthermore, the inclusion is strict if there exists $\theta \in \Theta$ such that $u_0(\theta; \pi) > v_o(\theta)$;*
2. *if the types are uniformly distributed ($f \equiv (\overline{\theta} - \underline{\theta})^{-1}$) the spread narrows, i.e. $\mathcal{S}_o \geq \mathcal{S}_m$;*



(a) Indirect Utilities (b) Lagrange Multiplier

Figure 3: An example without exclusion

3. the type-wise welfare increases, i.e. $v_o(\theta) \leq v(\theta; \pi)$ *for all* $\theta \in \Theta$.

Remark 2.10. *The results of Theorem 2.9 are borne by some of the empirical literature. For instance, Gresse (2006) concludes that*

> *"Looking at the relationship between CN trading and the DM liquidity, spreads are negatively related to CN-executed volumes."*

Buti et al. (2011) also provide evidence that high CN activity is associated with narrower spreads, although in their case no causality is concluded. Foley and Putniņš (2016) show that two-sided dark pools are beneficial, whereas the impact of one-sided dark pools is not clear and has an adverse-selection effect.

We finalize this section with two examples that showcase the results obtained thus far. Example 2.11 showcases that, in the simple case where the outside option is such that the dealer (only) excludes all high-enough (in absolute value) types, then the results of Theorem 2.9 follow trivially.

Example 2.11. *Let us revisit Example 2.2 with an extremely steep outside option that warrants exclusion, namely, for* $r_0 < r$ *let*

$$u_0(\theta) = \begin{cases} \infty, & \text{if } \theta \in [-r, -r_0) \bigcup (r_0, r]; \\ 0, & \text{otherwise.} \end{cases}$$

Recall that, for a given value Γ *of the Lagrange multiplier, the corresponding quantity is*

$$q(\theta; \Gamma) := 2\theta + r - 2r\Gamma.$$

In Example 2.2 the participation constraint does not bind for high types. In particular, $\gamma \equiv 0$ *on* $[-r, \underline{\theta}_0)$ *and to find the left-hand endpoint of the reserved set we set* $\Gamma = 0$ *and solve* $2\theta + r = 0$. *In* *the current setting, the participation constraint binds for* $\theta < -r_0$ *and the multiplier is constant on* $(-r_0, \underline{\theta}_0(\Gamma))$*, where*

$$\underline{\theta}_0(\Gamma) := -\frac{r}{2}\big[1-2\Gamma\big].$$

By construction, the choice of Γ *bears no weight on the trader types that are serviced to the left of* $\theta = -r_0$*, but only on how many additional low types benefit from the presence of the outside option. By integrating* $q(\theta;\Gamma)$ *and noting that the corresponding indirect-utility function* $v(\cdot;\Gamma)$ *must satisfy* $v(\underline{\theta}_0(\Gamma);\Gamma) = 0$*, we have, for* $\theta \in [-r_0, \underline{\theta}_0(\Gamma)]$

$$v(\theta;\Gamma) = \theta^2 + \theta r\big[1-2\Gamma\big] + \frac{r^2}{4}\big[1-2\Gamma\big]^2.$$

Given that the indirect-utility function also satisfies $v(\theta;\Gamma) = \theta q(\theta;\Gamma) - \tau(\theta;\Gamma)$*, we have that the DM on* $[-r_0, \underline{\theta}_0(\Gamma)]$ *is described by the quantity-price pairs* $\big(q(\theta;\Gamma), \theta^2 - \frac{r^2}{4}\big[1-2\Gamma\big]^2\big)$*. As a consequence, the per-type profit is*

$$\Pi(\theta;\Gamma) := -\theta^2 - \frac{3}{4}r^2\big[1-2\Gamma\big]^2 - 2\theta r\big[1-2\Gamma\big],$$

where the third term on the right-hand side is positive and dominates the first two. Finally, we have that each choice of Γ *results in the dealer obtaining the aggregate profits from negative types*

$$P(\Gamma) := \frac{1}{2r}\int_{-r_0}^{\underline{\theta}_0(\Gamma)} \Pi(\theta;\Gamma)d\theta.$$

The mapping $\Gamma \mapsto P(\Gamma)$ *is strictly concave and the first-order conditions yield that it is maximized at* $\Gamma = (r-r_0)/(2r)$*. As a result* $\underline{\theta}_0(\Gamma) = -r_0/2$ *and* $v(\theta;\Gamma) = \theta^2 + r_0\theta + r_0^2/4$*, which correspond to the boundary of the reserved set and the indirect-utility function for negative trader types in the problem without a CN on* $[-r_0, r_0]$*.*

Example 2.12. *We stay with the basic setup of Examples 2.3 and 2.8, but now assume that* $u_0(\theta;\pi) = \left(\frac{1-\pi_+}{3}\theta^{6/5} - 0.001\right)_+$ *for* $\theta \geq 0$ *and* $u_0(\theta;\pi) \equiv 0$ *otherwise. For any type* θ *such that* $u_0(\theta) > 0$ *it holds that*

$$\big(q_c(\theta), \tau_c(\theta)\big) = \Big(\frac{2}{5}(1-\pi_+)\theta^{1/5}, \frac{2}{5}(1-\pi_+)\theta^{6/5} + \frac{1}{25}(1-\pi_+)^2\theta^{2/5} - \big(\frac{1}{3}(1-\pi_+)\theta^{6/5} - 0.001\big)_+\Big).$$

We assume $\pi = (0, 1/2)$*. The first thing to notice is that the dealer's per-type profit for offering* $(q_c(\theta), \tau_c(\theta))$*, i.e.* $\tau_c(\theta) - C(q_c(\theta)) = \theta^{6/5}/30 - \theta^{2/5}/100 + 0.001$*, is negative for types* $\theta \in (0.0035, 0.1667)$*. On the other hand, the inequality* $u_0(\theta; 1/2) \geq 0$ *only holds for* $\theta \geq 0.014$*. Combining both arguments we see that* $\Theta_e(\pi) \subset (0.014, 0.1667)$*. Next we observe that the inequality*

$$l(\theta, 1) = K^{-1}\Big(\theta - \frac{1-F(\theta)}{f(\theta)}\Big) \geq \frac{\sqrt[5]{\theta}}{5}$$

holds for all $\theta \in [0.4761, 1]$*. As a consequence, we have that profitable matching may occur on the interval* $(0.1667, 0.4761)$*, over which* $q(\theta) = q_c(\theta)$ *and* $\mathbb{C}\big(q(\theta)\big) = C\big(q(\theta)\big)$*. Furthermore,* *Proposition 2.6 implies that the corresponding indirect utility function is differentiable at* $\theta = 0.4761$. *In order to obtain* $v(\theta;\pi)$ *for* $\theta \in [0.4761, 1]$, *we integrate* $l(\cdot, 1)$ *and determine the corresponding integration constant* c *by equating*

$$2\int_0^{0.4761} \left(\frac{3\theta^2 - 6\theta + 2}{2\theta - 3}\right) d\theta + c = \frac{1}{6}(0.4761)^{6/5} - 0.001.$$

We know from the example without a CN that $\gamma(t) = 0$ *for* $\theta \in [-1, -0.423)$. *On* $[-0.423, 0)$ *the multiplier must satisfy*

$$K^{-1}\Big(\theta - \frac{\gamma(\theta) - F(\theta)}{f(\theta)}\Big) = 0,$$

which results in $\gamma(\theta) = (3\theta^2 + 6\theta + 2)/4$ *on the said interval. What remains to be determined is* $\overline{\theta}_0$ *and* $\gamma(\overline{\theta}_0)$. *To this end, we define the family of functions* $v(\cdot;\Gamma)$ *such that* $v'(\theta;\Gamma) = l(\theta, \Gamma)$ *whenever this quantity is positive and* $v(\theta;\Gamma) = 0$ *for* $\theta \in [0, \theta(\Gamma)]$, *where* $\theta(\Gamma)$ *is the solution to the equation* $l(\theta, \Gamma) = 0$. *As* $\gamma(0) = 0.5$, *we have that* $\Gamma > 0.5$.[17] *In fact,* $\Gamma = \gamma(\overline{\theta}_0) = 0.5105$, $\overline{\theta}_0 = 0.007$ *and the intersection of* $v(\cdot;\Gamma)$ *and* $u_0(\cdot;1/2)$ *occurs at* $\theta = 0.0159$.

Summarizing, the types on $[-1, -0.423) \cup (0.007, 0.0159] \cup (0.1667, 1]$ *are fully serviced, those on* $[-0.423, 0.007]$ *are reserved and the ones that lie on* $(0.0159, 0.1667)$ *are excluded. The left-hand side of the spread is the same as in the example without a CN, whereas the right-hand side is* $t(0_+) = 0.0281$. *This is significantly smaller than in Example 2.3.*

Determining $\gamma(\theta)$ *on* $(0, 0.007]$ *is relatively simple, as we again must solve* $l(\theta, \gamma(\theta)) = 0$, *which results in* $\gamma(\theta) = (-3\theta^2 + 6\theta + 2)/4$. *Finally, in order to determine* γ *on* $\Theta_e(\pi)$ *we must rewrite the virtual surplus using* $\mathbb{C}(q(\theta)) = \tau_c(\theta)$, *which results in*

$$\mathbb{C}(q) = (5^5/6)q^6 - (1/4)q^2 + 0.001.$$

The pointwise maximization of the resulting virtual surplus must equal $q_c(\theta) = \sqrt[5]{\theta}/5$. *After some lengthy arithmetic that we choose to spare the reader from, we obtain*

$$\gamma(\theta) = F(\theta) - f(\theta)\left[5^5 q_c(\theta)^5 - \theta\right] = F(\theta) \quad \text{for } \theta \in \Theta_e(\pi).$$

Finally, in the profitable-matching region we solve $l(\theta, \gamma(\theta)) = \sqrt[5]{\theta}/5$ *so as to find the multiplier, which yields*

$$\gamma(\theta) = F(\theta) - f(\theta)\left[\frac{1}{10}\theta^{1/5} - \theta\right] \quad \text{for } \theta \in [0.1667, 0.4761).$$

or

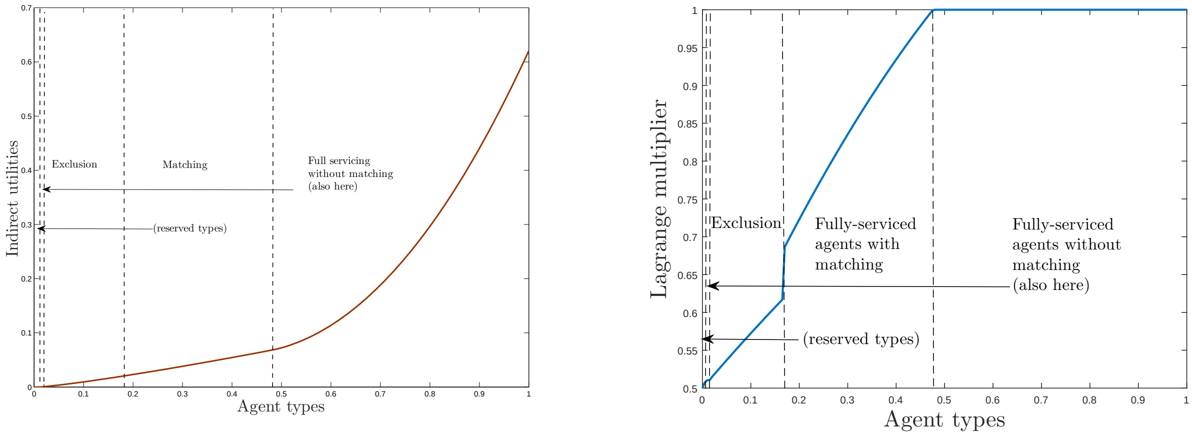
$$\gamma(\theta) = \frac{1}{10}\theta^{1/5} \cdot \frac{2\theta - 3}{4} - \frac{3\theta^2 - 6\theta - 2}{4} \quad \text{for } \theta \in [0.1667, 0.4761).$$

Observe that, in contrast with Example 2.8, here $\gamma(\theta) = 1$ *for types that are strictly smaller than one. This means that the rightmost types do not profit from the introduction of the CN via*

[17] Pasting when passing from servicing to excluding need not be smooth.

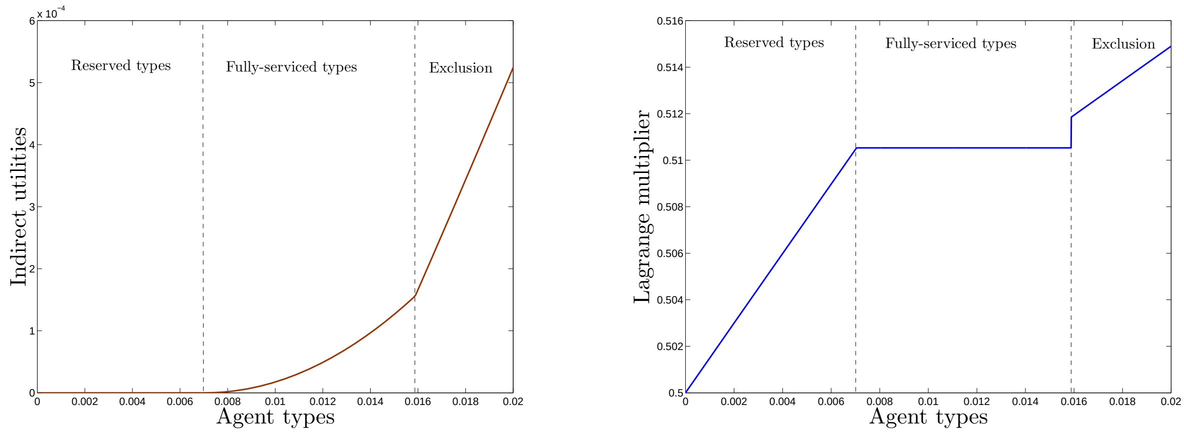
changes in the quantities they are offered, but rather from changes in the corresponding prices. Intuitively speaking this has to do with how steep the outside option is for large types and, as a consequence, whether or not it is matched over a non-trivial interval.

We present in Figure 4(a) the indirect utilities for positive types (the ones for negative ones being the same as in Figure 1(a)). The values of γ have been plotted in Figure 4(b). In Figure 5 we provide a magnification around small values of θ so as to highlight the switching between reservation, full servicing and exclusion. Observe the jump of the Lagrange multiplier at the boundary between fully-serviced and excluded types (Figure 5(b)) and between excluded and matched ones (Figure 4(b)).



(a) Indirect Utilities (b) Lagrange Multiplier

Figure 4: An example with exclusion



(a) Indirect Utilities (b) Lagrange Multiplier

Figure 5: An example with exclusion (magnified)

We revisit this example in the upcoming section, where we look into the existence of equilibrium prices in the CN.

3. An equilibrium price in the crossing network

Motivated by the fact that prices in CNs are obtained from those in a primary venue, it is natural to assume that pricing in the DM has an impact on the pricing schedule π. For example, trading in the CN could take place at the best-bid and best-ask prices of the primary market. We analyze such an example, within a portfolio-liquidation framework, in Section 4.

The pecuniary interaction between the DM and the CN, however, is not unidirectional: the dealer anticipates the effect that his choice of book structure has on the CN. Our main focus is the impact of the CN on the spread in the DM. Specifically, if we denote by $t(0;\pi) := \big(t(0-;\pi), t(0+;\pi)\big)$ the best bid-ask prices in the DM for a given CN price schedule π, then we call π^* an *equilibrium price* if $\pi^* = t(0;\pi^*)$. In this section we analyze the existence of an equilibrium price π^*.

We make the following natural assumption on the impact of π on the traders' outside option.

Assumption 3.1. *Let $\pi_1 \leq \pi_2$, where "$\leq$" is the lexicographic order in $\mathbb{R}^2$, then for all $\theta \in \Theta$ it holds that $u_0(\theta;\pi_1) \geq u_0(\theta;\pi_2)$. Furthermore, we assume that there exists $\left(\underline{\pi}_-, \overline{\pi}_+\right) \in \mathbb{R}^2$ such that $\mathrm{w}(\cdot;\pi) \leq 0$ for all (π_-,π_+) such that $\pi_- \leq \underline{\pi}_-$ and $\overline{\pi}_+ \leq \pi_+$.*

Observe that, from Assumption 3.1, there is no loss of generality in assuming that π^* belongs to some closed and bounded subset of $\mathbb{R}^2$, which we denote by Π. As a consequence we have that $t(0;\cdot) : \Pi \to \Pi$.

We are now ready to state of our final main result, whose proof can be found in Appendix C.

Theorem 3.2. *If types are uniformly distributed, then the mapping $\pi \mapsto t(0;\pi)$ has a fixed point.*

Summarizing, we have that the dealer can correctly anticipate the movements in prices in the CN when he designs the optimal pricing schedule for the DM. Furthermore, the presence of the CN is beneficial in terms of liquidity, market participation and the traders' welfare.

Remark 3.3. *The requirement of uniformly distributed types can be relaxed to the extent that if f and K are such that Conditions* (B1) *are satisfied, then the required monotonicity properties still apply. Unfortunately, these conditions cannot be verified ex-ante because they include the end points of the set of reserved traders.*

Example 3.4. *Let us go back to Example 2.12 (with exclusion), but introduce the feedback loop between the DM and the CN through the iteration $\pi_{i+1} = t(0;\pi_i)$. We initialize the recursion by setting $\pi_0 = (0, 1/2)$ and $\kappa = 0.001$, which are the parameters in the aforementioned example.*

We observe a very swift convergence. Indeed, it takes only four iterations to reach $\|v(\cdot;\pi_i) - v(\cdot;\pi_{i+1})\|_\infty \leq 10^{-5}$ and the indirect-utility functions in the third and fourth iteration are almost indistinguishable. The equilibrium price is $\pi^* = (0, 0.015)$. We present in Figure 6 the plots of the first four iterates. It is evident that each iteration results in a smaller set of reserved traders and in a higher indirect utility for all types. The spreads, the right endpoints of the reserved

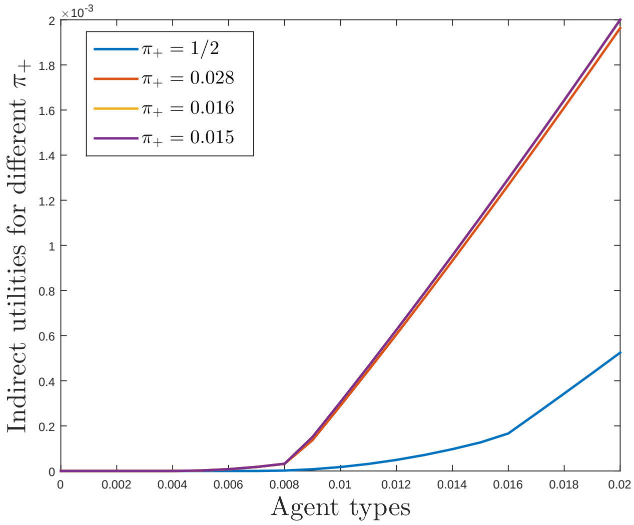


Figure 6: The indirect-utility functions corresponding to the iteration $\pi_{i+1} = t(0; \pi_i)$.

regions, the Lagrange multipliers at the right endpoint of the reserved regions and the exclusion regions are provided in Table 1. It is interesting to observe that, as the spread decreases to its equilibrium level, the number of trader types that are reserved decreases and the sets of excluded types grow (in terms of inclusions). This last fact obeys the fact that, when the traders have a more attractive outside option, it is harder for the dealer to match it profitably.

Table 1: The numbers of the feedback loop

π_+	Θ_o	Γ	$\Theta_e(\pi_+)$
1/2	[-0.423,0.0070]	0.5105	[0.0159, 0.1667]
0.0281	[-0.423,0.0040]	0.5061	[0.0083, 0.4872]
0.0161	[-0.423,0.0040]	0.5060	[0.0082, 0.4954]
0.0158	[-0.423,0.0040]	0.5060	[0.0082, 0.4955]

4. Portfolio liquidation and dark-pool trading

In this section we present an application of our methodology to portfolio liquidation. We assume that the market participants' aim is to liquidate part of their current holdings on some traded asset. The sizes of the traders' portfolios are heterogeneous and saying that a trader's type is θ means that she is short θ shares of the asset prior to trading (i.e. $\theta < 0$ implies the trader has a long position on the stock). We set $\Theta = [-1, 1]$ and $f \equiv 1/2$. If a trader of type θ trades q shares for τ dollars, his utility is

$$\hat{u}(\theta, q) - \tau := -\alpha(\theta - q)^2 - \tau,$$

where $0 < \alpha$ denotes the traders' (homogeneous) sensitivity towards inventory holdings. Notice that $-\alpha\theta^2$ is the type-dependent reservation utility of a trader of type θ. If we "normalize" the said utility to zero, we may write

$$u(\theta, q) - \tau = \underbrace{2\alpha q}_{\psi_1(q)}\,\theta \underbrace{-\alpha q^2}_{\psi_2(q)} - \tau. \tag{5}$$

In this example the crossing network takes the form of a *dark pool* (DP for short). Choosing to trade in the latter entails two kinds of costs for the traders: On the one hand, there is a direct, fixed cost $\kappa > 0$ of engaging in dark-pool trading. On the other hand, execution in the DP is not guaranteed. We denote by $p \in [0, 1]$ the probability that an order is executed where we assume for simplicity that the probability of order execution is independent of the order size. Pricing in the DP is linear. Namely, for given ask and bid execution prices $\pi = (\pi_a, \pi_b)$, the utility that a trader of type θ extracts from submitting an order of q shares to be traded in the DP is

$$p\big[(2\theta\alpha - \pi_j)q - \alpha q^2\big] - \kappa, \quad \text{with } \ j = a \ \text{ if } \ \theta \le 0 \ \text{ and } \ j = b \text{ otherwise,}$$

where again we have normalized reservation utilities to zero. We make the natural assumption that *maximal trades* cannot exceed the position that the corresponding agents hold. In particular, after trading in the DP, no one has gone from holding a long position to holding a short one and vice versa. The problem of optimal submission to the DP for a θ-type trader is

$$\max_q \Big\{p\big[(2\theta\alpha - \pi_j)q - \alpha q^2\big]\Big\},$$

with $j = a, b$ depending on the sign of θ, as above. Absent our assumption on maximal trade size this would yield the optimal submission level

$$\tilde{q}_d(\theta) = \theta - \frac{\pi_j}{2\alpha}.$$

With restricted trades, however, we have that

$$q_d(\theta) = \begin{cases} \theta, & \text{if } \theta \le 0; \\ 0, & \text{if } \theta \in (0, \frac{\pi_b}{2\alpha}); \\ \theta - \frac{\pi_b}{2\alpha}, & \text{if } \theta \in [\frac{\pi_b}{2\alpha}, 1]. \end{cases}$$

Note there is no loss of generality in assuming that $\pi < 2\alpha$, which establishes a link between trader impatience and the DP execution price. We obtain that opting for the DP results in a trader of type θ enjoying the expected utility

$$\mathrm{w}(\theta; \pi) = \begin{cases} (2p - 1)\alpha\theta^2 - p\pi_a\theta - \kappa, & \text{if } \theta \le 0; \\ -\kappa, & \text{if } \theta \in (0, \frac{\pi_b}{2\alpha}); \\ \alpha p \left(\theta - \frac{\pi_b}{2\alpha}\right)^2 - \kappa, & \text{if } \theta \in [\frac{\pi_b}{2\alpha}, 1]. \end{cases}$$

In order for $\mathrm{w}(\cdot;\pi)$ to be convex, it must hold that $p > 1/2$. In other words, the probability of execution in the DP cannot be too small. Under this condition Assumption 1.1 is clearly satisfied.

Remark 4.1. *Observe that, even though the optimal submission level q_d is independent of the probability of execution p, that is not the case for the traders' expected utilities of submitting an order to the DP. Given that the agents' outside options depend on the value of p, so does the spread, as we show below.*

Finally, we assume that the dealer's cost of unwinding a portfolio of size q is $C(q) = \beta q^2$ where $\beta > 0$.

4.1. The dealer market without a dark pool

Unlike trading in the DP, we do not make a maximal-trade-size assumption when agents engage the dealer. Given that trader types are uniformly distributed, Condition (3) on the Hazard rates is trivially satisfied. Therefore, in the absence of a DP, the dealer's optimal choices of quantities are, for negative types

$$l(\theta, 0) = K^{-1}\Big(\theta + \frac{F(\theta)}{f(\theta)}\Big) = \frac{\alpha}{\alpha+\beta}\big(2\theta + 1\big)$$

and for positive types

$$l(\theta, 1) = K^{-1}\Big(\theta - \frac{1-F(\theta)}{f(\theta)}\Big) = \frac{\alpha}{\alpha+\beta}\big(2\theta - 1\big),$$

where the boundary of Θ_0 is given by

$$l(\theta, 0) = 0 \Rightarrow \underline{\theta}_0 = -\frac{1}{2} \quad \text{and} \quad l(\theta, 1) = 0 \Rightarrow \overline{\theta}_0 = \frac{1}{2}.$$

Recall that from the Envelope Theorem and the convexity of the indirect-utility function we have that $v(\theta) = \theta\, v'(\theta) + \psi_2 \circ \psi_1^{-1}\big(v'(\theta)\big) - \tau(\theta)$ and $v'(\theta) = \psi_1\big(q(\theta)\big) = 2\alpha q(\theta)$. The indirect-utility function is then

$$v(\theta) = \begin{cases} \frac{2\alpha^2}{\alpha+\beta}(\theta^2 + \theta) - \frac{\alpha^2}{4(\alpha+\beta)}, & \theta \le \underline{\theta}_0; \\ \frac{2\alpha^2}{\alpha+\beta}(\theta^2 - \theta) - \frac{\alpha^2}{4(\alpha+\beta)}, & \theta \ge \overline{\theta}_0. \end{cases}$$

When it comes to the spread, observe that $q' \equiv \frac{2\alpha}{\alpha+\beta}$, $\phi_1 = \psi_1'(0) = 2\alpha$ and $\phi_2 = \psi_2'(0) = 0$, which from Expression (4) yields

$$[t(0-), t(0+)] = \frac{4\alpha^2}{\alpha+\beta}\big[-\frac{1}{2}, \frac{1}{2}\big].$$

Below we analyze how the spread changes with the introduction of the DP.

4.2. The impact of a dark pool

Recall that, combining the no-trade option and the possibility of trading in the DP, the trader's outside option is

$$u_0(\theta;\pi) = \max\big\{\mathrm{w}(\theta;\pi), 0\big\} \qquad \theta \in \Theta.$$

We denote by ϵ_1 and ϵ_2 the negative and positive roots of the equation $\mathrm{w}(\theta;\pi) = 0$, respectively. In other words, these are the boundary types where the agents' outside option yields positive expected utility. We first take an exogenous execution price π and determine, for each $\theta \in \Theta$, what is the quantity-price pair $\big(q_c(\theta;\pi), \tau_c(\theta;\pi)\big)$ that the dealer must offer so as to match a DP with execution price π. Using the relation $2\alpha q_c(\theta;\pi) = u_0'(\theta;\pi)$ we obtain

$$q_c(\theta;\pi) = \begin{cases} (2p-1)\theta - p\frac{\pi_a}{2\alpha}, & \text{if } \theta \le \epsilon_1; \\ p\left(\theta - \frac{\pi_b}{2\alpha}\right), & \text{if } \theta \ge \epsilon_2. \end{cases}$$

and

$$\tau_c(\theta;\pi) = \kappa + 2\alpha q_c(\theta;\pi) - q_c(\theta;\pi)^2 - u_0(\theta;\pi).$$

For agents who participate in the DM, we again have their indirect utility function satisfies

$$\frac{v'(\theta)}{2\alpha} = l\big(\theta, \gamma(\theta)\big). \tag{6}$$

In order to determine the spread in the presence of the DP we must determine $\underline{\theta}_{0,m}(\pi)$ and $\overline{\theta}_{0,m}(\pi)$, the left and right endpoints of the set of reserved types in the presence of the DP, respectively, together with $\gamma\big(\underline{\theta}_{0,m}\big)$ and $\gamma\big(\overline{\theta}_{0,m}\big)$. For an arbitrary $\Gamma \in [0,1]$ we have

$$l(\theta,\Gamma) = \frac{\alpha}{\alpha+\beta}\big[2\theta + 1 - 2\Gamma\big].$$

Indexed by Γ, the candidates for $\underline{\theta}_{0,m}(\pi)$ are then given by $\underline{\theta}_{0,m}(\Gamma;\pi) = \Gamma - 1/2$. As it must hold that $\underline{\theta}_{0,m}(\Gamma;\pi) \le 0$, then $\Gamma \le 1/2$. Let $\widetilde{\theta}_m(\Gamma)$ be the first intersection to the left of $\underline{\theta}_{0,m}(\Gamma)$ of $v(\cdot;\Gamma)$ and $u_0(\cdot;\pi)$, then, integrating Expression (6) with $\gamma(\theta) = \Gamma$ we have that, on $[\widetilde{\theta}_m(\Gamma;\pi), \underline{\theta}_{0,m}(\Gamma;\pi)]$, the traders' indirect utility is given by

$$v(\theta;\Gamma) = \frac{2\alpha^2}{\alpha+\beta}\big(\theta^2 + (1-2\Gamma)\theta\big) + c_{1,m}, \tag{7}$$

where $c_{1,m}$ is determined by the equation

$$v\big(\underline{\theta}_{0,m}(\Gamma;\pi);\Gamma\big) = 0.$$

Unless the inequality $\Gamma \le 1/2$ is tight, in which case the types below $\widetilde{\theta}_m(\Gamma)$ are excluded, Proposition 2.6 implies that Γ must be chosen so as to satisfy the smooth-pasting condition

$u_0'\big(\widetilde{\theta}_m(\Gamma;\pi);\pi\big) = v'\big(\widetilde{\theta}_m(\Gamma;\pi);\pi\big)$, which is equivalent to

$$\widetilde{\theta}_m(\Gamma;\pi) = -\Big[\frac{2\alpha}{\alpha+\beta} - (2p-1)\Big]^{-1}\Big[\frac{\alpha}{\alpha+\beta}(1-2\Gamma) + \frac{p\pi}{2\alpha}\Big]. \tag{8}$$

Observe that, besides the requirement $\Gamma \geq 1/2$, the strategy to determine $\overline{\theta}_{0,m}(\pi)$ is exactly the same as for $\underline{\theta}_{0,m}(\pi)$, with $\widehat{\theta}_m(\Gamma;\pi)$ playing the role of $\widetilde{\theta}_m(\Gamma;\pi)$ for positive types. Summarizing, from Eq. (7) we observe that, if Γ_- and Γ_+ correspond to the optimal choices for the negative and positive endpoints of $\Theta_0(\pi)$, then

$$q'\big(\underline{\theta}_{0,m}(\Gamma_-;\pi)\big) = \frac{1}{2\alpha}v''\big(\underline{\theta}_{0,m}(\Gamma_-;\pi);\Gamma_-\big) = \frac{1}{2\alpha}v''\big(\overline{\theta}_{0,m}(\Gamma_+;\pi);\Gamma_+\big) = q'\big(\overline{\theta}_{0,m}(\Gamma_+;\pi)\big) = \frac{2\alpha}{\alpha+\beta}.$$

The spread is then

$$|t_m(0+) - t_m(0-)| = \frac{4\alpha^2}{\alpha+\beta}|\overline{\theta}_{0,m}(\Gamma_+;\pi) - \underline{\theta}_{0,m}(\Gamma_-;\pi)| \leq \frac{4\alpha^2}{\alpha+\beta}|\overline{\theta}_0 - \underline{\theta}_0|,$$

i.e. the presence of a dark pool narrows the spread in the DM. This effect is strict if $\widetilde{\theta}_m(\Gamma_-;\pi) > -1$ or $\widehat{\theta}_m(\Gamma_+;\pi) < 1$, i.e. as soon as the presence of the DP is nontrivial. This is because these conditions imply that either $\Gamma_- > 0$ or $\Gamma_+ < 1$ which readily implies

$$\underline{\theta}_{0,m}(\Gamma_-;\pi) = \Gamma_- - \frac{1}{2} > \underline{\theta}_0 \quad \text{or} \quad \overline{\theta}_{0,m}(\Gamma_+;\pi) = \Gamma_+ - \frac{1}{2} < \overline{\theta}_0.$$

Given that $u_0(\cdot;\pi)$ satisfies Assumption 3.1, Theorem 3.2 may be applied directly to guarantee the existence of an equilibrium price π^*.

Remark 4.2. *It follows from Eq.* (8) *that the optimal* Γ_- *and* Γ_+ *that determine the spread are (implicit) functions of the probability of execution* p*. Intuitively, a higher* p *leads to a more attractive outside option for higher types and, as a consequence, to a smaller spread. The determination of the optimal* $\Gamma_-(p)$ *and* $\Gamma_+(p)$*, however, must ensue through the maximization of the dealer's objective as a function of* Γ*, which cannot be done unless we specify the model's parameters.*

5. Conclusions

We have presented a hidden-information model to study the structure of the limit-order book of a dealer who provides liquidity to traders of unknown preferences. Furthermore, we have established a link between the traders' indirect-utility function and the bid-ask spread in the DM. Making use of the aforementioned link, we have studied how the presence of a type-dependent outside option impacts the spread of the DM, as well as the set of trader types who participate in the DM and their welfare. In particular, we have shown, in a portfolio-liquidation setting, that the presence of a dark pool results in a shrinkage of the spread in the DM. Finally, we have established that, under certain conditions, the feedback loop introduced by the impact that the spread has on the structure of the outside option leads to an equilibrium price.

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Appendices

Appendix A Existence of a solution to Problem $\mathcal{P}(\pi)$

In this appendix we prove the existence of a solution to the dealer's problem in the presence of a CN. Some of the arguments are somewhat standard, but we give them for completeness. We require the following technical assumption:

Assumption A.1. *The functions ψ_1, ψ_2 and C are such that $\psi_1'(q) > 0$ and $C(q) - \psi_2(q) \geq 0$ hold for all $q \in \mathbb{R}$ and $\widetilde{K}$ is strictly convex, coercive,*[18] *continuously differentiable and satisfies $\widetilde{K}'(0) = 0$.*

The first important result that we require is that the dealer's optimal choices lead to him never losing money on types that participate.

Proposition A.2. *If $(q^*, \tau^*) : \Theta \to \mathbb{R}^2$ is an optimal allocation, then for all participating types it holds that $\tau^*(\theta) - C\big(q^*(\theta)\big) \geq 0$.*

Proof. Assume the contrary, i.e. that the set

$$\widetilde{\Theta} := \big\{\theta \,|\, v(\theta;\pi) \geq u_0(\theta;\pi), \tau^*(\theta) < C\big(q^*(\theta)\big)\big\},$$

where $v(\theta;\pi) = u\big(\theta, q^*(\theta)\big) - \tau^*(\theta)$ has positive measure. Define a new pricing schedule via

$$\widetilde{\tau}(\theta) := \max\big\{\tau^*(\theta), C\big(q^*(\theta)\big)\big\}.$$

The incentives for types in $\widetilde{\Theta}^c$ do not change because their prices remain unchanged, whereas prices for others have increased. Profits corresponding to trading with types in $\widetilde{\Theta}$ increase to zero. As a consequence the dealer's welfare strictly increases, which violates the optimality of (q^*, τ^*). □

A consequence of Proposition A.2 is that, together with Assumption A.1, it allows us to restrict the admissible set of the dealer's problem to a compact one. We prove this in several steps.

Lemma A.3. *If $v : \Theta \to \mathbb{R}$ is a non-negative, convex function that solves $\mathcal{P}$, then $v(0) = 0$.*

Proof. Assume that $v \in \mathcal{C}$ solves $\mathcal{P}$ and $v(0) > 0$. This implies that $\psi_2\big(q(0)\big) - \tau(0) \geq 0$. Given that, from Assumption 1.1, a trader of type $\theta = 0$ has no access to a profitable outside option, then she participates. From Proposition A.2 it must then hold that $\tau(0) \geq C\big(q(0)\big)$ which in turn implies that $\psi_2\big(q(0)\big) \geq C\big(q(0)\big)$. This relation, however, can only hold for $q(0) = 0$, which implies that $\tau(0) = v(0) = 0$. □

[18] By coercive we mean that $\lim_{|q|\to\infty} C(q) = \infty$.

Lemma A.4. *If $v \in \mathcal{C}$ solves $\mathcal{P}$, then $|\partial v| \leq \overline{q}$.*

PROOF. From Assumption A.1 and the compactness of Θ we have that the mapping $q \mapsto i(\theta, v, q)$ tends to $-\infty$ as $|q| \to \infty$ uniformly on Θ for $v \geq 0$. From Proposition A.2 $i\big(\theta, v(\theta), v'(\theta)\big)$ must be non-negative for all participating types, which concludes the proof. □

As $\overline{q}$ could depend on π, we define

$$\mathcal{A}(\pi) := \{v \in \mathcal{C} \,|\, v \geq 0, v(0) = 0, |\partial v| \leq \overline{q}\}$$

as new admissibility set for problem $\mathcal{P}(\pi)$. The previous results show that if we replace $\mathcal{C}$ by $\mathcal{A}(\pi)$ in the definition of $\mathcal{P}(\pi)$, the solution to the problem does not change.

Corollary A.5. *The admissible set $\mathcal{A} \subset \mathcal{C}$ of Problem $\mathcal{P}$ is uniformly bounded and uniformly equicontinuous.*

PROOF. From Lemmas A.3 and A.4, a uniform bound for all $v \in \mathcal{A}$ is given by $\max_{\theta\in\Theta} \big\{u_0(\theta;\pi)\big\} + \overline{q}\|\Theta\|$. Lemma A.4 guarantees that for any $v \in \mathcal{A}$ it holds that $|\partial v| \leq \overline{q}$. In other words, $\mathcal{A}$ is composed of convex functions whose subdifferentials are uniformly bounded, hence $\mathcal{A}$ is uniformly equicontinuous. □

Notice that, when it comes to determining quantities and prices for trader types who do participate, Proposition A.2 results in the dealer having to solve the problem

$$\widetilde{\mathcal{P}}(\pi) := \begin{cases} \sup_{v\in\mathcal{A}} \int_\Theta \big(i\big(\theta, v(\theta), v'(\theta)\big)\big)_+ f(\theta)d\theta \\ \text{s.t. } v(\theta) \geq u_0(\theta;\pi) \text{ for all } \theta \in \Theta. \end{cases}$$

The last auxiliary result that we need is the following proposition, whose proof is a direct consequence of Fatou's Lemma, together with Lemmas A.3 and A.4.

Proposition A.6. *The mapping*

$$v \mapsto \int_\Theta \big(i(\theta, v(\theta), v'(\theta))\big)_+ f(\theta)d\theta$$

is upper semi-continuous in $\mathcal{A}$ with respect to uniform convergence.

We are now ready to prove existence of a solution to the principal's problem: Assume that $\mathcal{A}\bigcap\big\{v \in \mathcal{C}|v(\cdot) \geq u_0(\cdot;\pi)\big\}$ is non-empty and consider a maximizing sequence $\big\{\widetilde{v}_n\big\}_{n\in\mathbb{N}}$ of Problem $\widetilde{\mathcal{P}}(\pi)$. From Corollary A.5 we have that, passing to a subsequence if necessary, there exists $\widetilde{v} \in \mathcal{A}$ such that $\widetilde{v}_n \to \widetilde{v}$ uniformly. A direct application of Proposition A.6 yields that $\widetilde{v}$ is a solution to $\widetilde{\mathcal{P}}(\pi)$. To finalize the proof we must construct from $\widetilde{v}$ a solution to Problem $\mathcal{P}(\pi)$. To this end, let us define the sets

$$\Theta_- := \big\{\theta \in \Theta | i\big(\theta, \widetilde{v}(\theta), \widetilde{v}'(\theta)\big) < 0\big\} \quad \text{and} \quad \Theta_+ := \Theta_-^c.$$

It is well known that if a sequence of convex functions converges uniformly (to a convex function), then there is also uniform convergence of the derivatives wherever they exist, which is almost everywhere. This fact, together with the continuity of the mappings $\theta \mapsto \widetilde{v}(\theta)$ and $(\theta, v, q) \mapsto$

$i(\theta, v, q)$, implies that Θ_- is the union of a disjoint set of open intervals:

$$\Theta_- = \bigcup_{i=1}^{\infty} (a_i, b_i).$$

Define, for each $i \geq 1$,

$$\widetilde{v}_{a,i} := \inf \left\{ q | q \in \partial \widetilde{v}(a_i) \right\} \quad \text{and} \quad \widetilde{v}_{b,i} := \sup \left\{ q | q \in \partial \widetilde{v}(b_i) \right\}$$

and consider the support lines to graph$\{\widetilde{v}\}$ at a_i and b_i given by

$$l_i(\theta) = \widetilde{v}(a_i) + \widetilde{v}_{a,i}(\theta - a_i) \quad \text{and} \quad L_i(\theta) = \widetilde{v}(b_i) + \widetilde{v}_{b,i}(\theta - b_i),$$

respectively. Let $c_i \in (a_i, b_i)$ be, for each $i \geq 1$, the unique solution to the equation $l_i(\theta) = L_i(\theta)$ and define on $(a_i, b_i) =: \Theta_i$

$$v_i^*(\theta) := \begin{cases} l_i(\theta) & \theta \leq c_i; \\ L_i(\theta) & \theta > c_i. \end{cases}$$

Finally define

$$v^*(\theta) := \begin{cases} \widetilde{v}(\theta) & \theta \in \Theta_+; \\ v_i^*(\theta) & \theta \in \Theta_i, i \in \mathbb{N}, \end{cases}$$

then v^* is a solution to Problem $\mathcal{P}(\pi)$ and $\Theta_e(v^*) = \Theta_-$, which concludes the proof. □

Appendix B The impact of a CN on the DM

In order to prove Theorem 2.9, we require a result that guarantees that our notion of the spread is well defined in the presence of a CN. This could be loosely summarized by saying that the first (in terms of moving away from $\theta = 0$) types to earn positive utility trade in the DM.

Lemma B.1. *There exists $\epsilon = \epsilon(\pi)$ such that the types that belong to*

$$(\underline{\theta}_0(\pi) - \epsilon, \underline{\theta}_0(\pi)) \cup (\overline{\theta}_0(\pi), \overline{\theta}_0(\pi) + \epsilon)$$

are fully serviced.

Proof. Let us define $\hat{\theta} := \sup \left\{ \theta \in \Theta \mid u_0(\theta; \pi) = 0 \right\}$. If there exists $\eta > 0$ such that types on $(\hat{\theta}, \hat{\theta} + \eta)$ can be matched profitably, then the result follows either because $\overline{\theta}_0(\pi) < \hat{\theta}$ or because $\overline{\theta}_0(\pi) = \hat{\theta}$ and the types on $(\hat{\theta}, \hat{\theta} + \epsilon)$, for some $0 < \epsilon \leq \eta$, are fully serviced. Let us now assume that such an η does not exist, we claim then that $\overline{\theta}_0(\pi) < \hat{\theta}$ must hold. Proceeding by the way of contradiction, let us assume that $\overline{\theta}_0(\pi) = \hat{\theta}$ and that there exists $\delta > 0$ such that $(\hat{\theta}, \hat{\theta} + \delta) \subset \Theta_e(\pi)$. This configuration can be improved upon as follows: let $a > 0$ be such that $\hat{\theta} - a > 0$. By construction $l(\hat{\theta} - a, \gamma(\hat{\theta} - a)) = 0$. Let us fix $\gamma(\theta) \equiv \gamma(\hat{\theta} - a) =: \Gamma(a)$ for $\theta \in (\hat{\theta} - a, \theta_a)$, where θ_a the solution to $v_a(\theta) = u_0(\theta; \pi)$ on $(\hat{\theta} - a, \overline{\theta}]$ if it exists or $\theta_a = \overline{\theta}$ otherwise, given that we denote by v_a the indirect-utility function corresponding to setting $\gamma(\theta) \equiv \Gamma(a)$ for $\theta \in (\hat{\theta} - a, \overline{\theta})$. In particular $\theta_a > \hat{\theta}$ and $l(\theta, \Gamma(a)) > 0$ for $\theta \in (\hat{\theta} - a, \theta_a)$.

We now have that types $\theta \in (\hat{\theta} - a, \theta_a)$ are fully serviced. By Assumption 1.1, $v'_a(\hat{\theta} - a) = 0 < u'_0(\hat{\theta}; \pi)$; therefore, there exists $a_1 > 0$ such that for all $a \leq a_1$ it holds that $\theta_a < \hat{\theta} + \delta$. If we could show that there exists $a \leq a_1$ such that the dealer could offer types in $(\hat{\theta} - a, \theta_a)$ the quantities $q_a(\theta) = l(\theta, \Gamma(a))$ at a profit, we would contradict the optimality of $\overline{\theta}_0(\pi)$ and the proof would be finalized, as incentives above θ_a would not be distorted and the dealer's profits would strictly increase. In order to do so, observe that the dealer's typewise profit when offering $q_a(\theta)$ is

$$P(\theta) := \theta\psi_1(q_a(\theta)) + \psi_2(q_a(\theta)) - v_a(\theta) - C\big(q_a(\theta)\big).$$

In particular, $P(\hat{\theta} - a) = 0$ and

$$\begin{aligned} P'(\hat{\theta} - a) &= \psi_1(q_a(\hat{\theta} - a)) + (\hat{\theta} - a)\psi_1'(q_a(\hat{\theta} - a))q_a'(\hat{\theta} - a) + v_a'(\hat{\theta} - a) \\ &\quad - \widetilde{C}'\big(q_a(\hat{\theta} - a)\big)q_a'(\hat{\theta} - a) \\ &= \psi_1(0) + (\hat{\theta} - a)\psi_1'(0)q_a'(\hat{\theta} - a) + v_a'(\hat{\theta} - a) - \widetilde{C}'\big(0\big)q_a'(\hat{\theta} - a) \\ &= (\hat{\theta} - a)\psi_1'(0)q_a'(\hat{\theta} - a). \end{aligned}$$

The step from the second to the third equality follows, because by construction $v_a'(\hat{\theta} - a) = 0$; by assumption $\psi_1(0) = 0$ and, from Assumption A.1, $\widetilde{C}'\big(0\big) = 0$. Furthermore, given that ψ_1 is strictly increasing and $q_a'(\hat{\theta} - a) > 0$, then $P'(\hat{\theta} - a) > 0$. Therefore, there exists $b > 0$ such that $P(\theta) > 0$ if $\theta \in (\hat{\theta} - a, \hat{\theta} - a + b)$. As a consequence, if $a < a_1$ is small enough, then $P(\theta) > 0$ for $\theta \in (\hat{\theta} - a, \theta_a)$, as required. □

We are now ready to prove our second main result:

Proof of Theorem 2.9: (1) Observe that if π is such that $\big(\underline{\theta}_0(\pi), \overline{\theta}_0(\pi)\big) = \Theta_0(\pi) \subset \Theta_o$, then the result follows immediately from Lemma B.1. If we revert the inclusion, two situations are possible, as the addition of the CN-constraint to Problem $\mathcal{P}_o$ may or may not bind for some types. The latter case being trivial, let us look at the case where there is a point $\theta_a > \overline{\theta}_0$ on which it holds that $v_o(\theta_a) = u_0(\theta_b; \pi)$ and such that $v_o(\theta) > u_0(\theta; \pi)$ for $\theta < \theta_a$ and vice versa for $\theta > \theta_a$. The Lagrange multiplier γ_m is active on $(\theta_a, \overline{\theta}]$, which implies that $\gamma_m(\theta_a) < 1$. We know from Jullien (2003), p. 9, that for all θ such that $l(\theta, \Gamma) > 0$, the latter is decreasing in Γ. As a consequence, the root of the equation

$$K^{-1}\Big(\theta + \frac{F(\theta) - \gamma_m(\theta_a)}{f(\theta)}\Big) = 0$$

is strictly smaller than that of $l(\theta, 1) = 0$, which yields the desired result.

(2) Let us denote by $t_o(0-)$ and $t_o(0+)$ the best bid and ask prices without the presence of a CN and by $t_m(0-)$ and $t_m(0+)$ the corresponding marginal prices with one; thus,

$$t_o(0-) = q_o'(\underline{\theta}_{0,o-})\big(\underline{\theta}_{0,o}\phi_1 + \phi_2\big) \text{ and } t_o(0+) = q_o'(\overline{\theta}_{0,o+})\big(\overline{\theta}_{0,o}\phi_1 + \phi_2\big)$$

and

$$t_m(0-) = q_m'(\underline{\theta}_{0,m-})\big(\underline{\theta}_{0,m}\phi_1 + \phi_2\big) \text{ and } t_m(0+) = q_m'(\overline{\theta}_{0,m+})\big(\overline{\theta}_{0,m}\phi_1 + \phi_2\big).$$

From Part (1) we know that $\underline{\theta}_{0,o} \leq \underline{\theta}_{0,m}$ (both negative) and $\overline{\theta}_{0,m} \leq \overline{\theta}_{0,o}$ (both positive) and, given that ϕ_1 and ϕ_2 do not depend on the presence of the CN, all we have left to do is show that

$$q_m'(\underline{\theta}_{0,m-}) \leq q_o'(\underline{\theta}_{0,o-}) \text{ and } q_m'(\underline{\theta}_{0,m+}) \leq q_o'(\underline{\theta}_{0,o+}).$$

Using the well-known relation $(f^{-1})'(a) = 1/f'(f^{-1}(a))$ we have that

$$\begin{aligned} q_m'(\underline{\theta}_{0,m-}) &= \frac{1}{K'\Big(K^{-1}\big(\underline{\theta}_{0,m} - \frac{\gamma(\underline{\theta}_{0,m-})-F(\underline{\theta}_{0,m})}{f(\underline{\theta}_{0,m})}\big)\Big)} \frac{d}{d\theta}\big(\theta - \frac{\gamma(\theta)-F(\theta)}{f(\theta)}\big)\Big|_{\theta=\underline{\theta}_{0,m-}} \\ &= \frac{1}{K'\big(q_m(\underline{\theta}_{0,m})\big)} \frac{d}{d\theta}\big(\theta - \frac{\gamma(\theta)-F(\theta)}{f(\theta)}\big)\Big|_{\theta=\underline{\theta}_{0,m-}} \\ &= \frac{1}{K'(0)}\Big(1 - \frac{d}{d\theta}\big(\frac{\gamma(\underline{\theta}_{0,m-})-F(\theta)}{f(\theta)}\big)\Big)\Big|_{\theta=\underline{\theta}_{0,m}}, \end{aligned}$$

where we have used the fact that γ is constant on $(\underline{\theta}_{0,m} - \delta, \underline{\theta}_{0,m})$ for some $\delta > 0$. We may proceed analogously for the other three quantities. We have to show that

$$\begin{aligned} \frac{1}{K'(0)}\frac{d}{d\theta}\Big(\frac{\gamma(\underline{\theta}_{0,m-})-F(\theta)}{f(\theta)}\Big)\Big|_{\theta=\underline{\theta}_{0,m}} &\geq \frac{1}{K'(0)}\frac{d}{d\theta}\Big(\frac{-F(\theta)}{f(\theta)}\Big)\Big|_{\theta=\underline{\theta}_{0,o}} \\ \frac{1}{K'(0)}\frac{d}{d\theta}\Big(\frac{\gamma(\overline{\theta}_{0,m+})-F(\theta)}{f(\theta)}\Big)\Big|_{\theta=\overline{\theta}_{0,m}} &\geq \frac{1}{K'(0)}\frac{d}{d\theta}\Big(\frac{1-F(\theta)}{f(\theta)}\Big)\Big|_{\theta=\overline{\theta}_{0,o}}, \end{aligned} \tag{B1}$$

which hold with equality under the assumption that $f \equiv (\overline{\theta} - \underline{\theta})^{-1}$.

(3) If follows from Part (1) that, if θ participates in the presence of the CN, then $q_o(\theta) \leq q_m(\theta)$. Assume now that the inequality $v_o(\theta) > v(\theta;\pi)$ holds for all θ in a non-empty interval (θ_1, θ_2) and $v_o(\theta_1) = v(\theta_1;\pi)$ and $v_o(\theta_2) = v(\theta_2;\pi)$. By the convexity of v_o and $v(\cdot;\pi)$, this would imply the existence of $\theta_3 \in (\theta_1, \theta_2)$ such that $v_o'(\theta) > v'(\theta;\pi)$ holds almost surely in (θ_1, θ_3). However $v_o'(\theta) = \psi_1(q_o(\theta))$, $v'(\theta;\pi) = \psi_1(q_m(\theta))$ and ψ_1 is strictly increasing; hence, this would imply that $q_o(\theta) > q_m(\theta)$ for almost all $\theta \in (\theta_1, \theta_3)$, which is a contradiction. ☐

Appendix C The existence of an equilibrium price.

The restriction of possible equilibrium prices to Π, together with Assumptions 1.1 and 3.1, yields the next result.

Lemma C.1. *There exists a non-empty interval* $[\epsilon_1, \epsilon_2] \subset \Theta$ *such that*

1. $0 \in (\epsilon_1, \epsilon_2)$;
2. $u_0(\theta;\pi) = 0$ *for all* $\theta \in [\epsilon_1, \epsilon_2]$ *and all* $\pi \in \Pi$.

In the sequel we make use of the results obtained in Section 2.2 to show that the mapping $\pi \mapsto t(0;\pi)$ has the required monotonicity properties so as to use the following result (see, e.g. Aliprantis and Border (2007)):

Theorem C.2. *(Tarski's Fixed Point Theorem) Let* $(X, \leq)$ *be a non-empty, complete lattice. If* $f : X \to X$ *is order preserving, then the set of fixed points of* f *is also a non-empty, complete lattice.*

Proof of Theorem 3.2. Lemmas B.1 and C.1 guarantee that we have a well-defined spread; thus, we may decompose the analysis of the mapping $\pi \mapsto t(0;\pi)$ into that of the mappings $\pi_- \mapsto t(0-;\pi_-)$ and $\pi_+ \mapsto t(0+;\pi_+)$. In other words, for a given price π, the dealer's optimal response to $u_0(\cdot;\pi)$ is, modulo a normalization of γ, equivalent to the combination of his actions towards negative and positive types separately. We concentrate on the existence of a fixed point of the mapping $\pi_+ \mapsto t(0+;\pi_+)$.

From Assumption 3.1 we have that if $\pi_{1+} < \pi_{2+}$, then $u_0(\theta; \pi_{1+}) > u_0(\theta; \pi_{2+})$ for all $\theta > 0$. If for $i = 1, 2$ it holds that $u_0(\theta; \pi_{i+}) < v_o(\theta)$ for all $\theta > 0$, then $v(\theta; \pi_{1+}) = v(\theta; \pi_{2+})$ on the same domain and $t(0_+; \pi_{1+}) = t(0_+; \pi_{2+})$. Next assume that $u_0(\theta; \pi_{i+}) \geq v_o(\theta)$ on a subset Θ_i of $(0, \overline{\theta}]$, for $i = 1, 2$. Given that $u_0(\theta; \pi_{1+}) > u_0(\theta; \pi_{2+})$ for all $\theta > 0$, then $\overline{\theta}(\pi_1) < \overline{\theta}(\pi_2)$ and the first point $\widetilde{\theta}_1$ such that $v(\theta; \pi_{1+}) = u_0(\theta; \pi_{1+})$ holds satisfies $\widetilde{\theta}_1 < \widetilde{\theta}_2$, where the latter is the analogous to $\widetilde{\theta}_1$ in the presence of $u_0(\theta; \pi_{2+})$. The existence of $\widetilde{\theta}_1$ and $\widetilde{\theta}_2$ is guaranteed by the fact that in both cases the indirect-utility functions intersect the corresponding outside options. Arguing as in the proof of Theorem 2.9, Part (2), this also implies that $\overline{\theta}_0(\pi_1) < \overline{\theta}_0(\pi_2)$; hence $t(0_+; \pi_{1+}) < t(0_+; \pi_{2+})$. In other words, the mapping $\pi_+ \mapsto t(0_+; \pi_+)$ is order-preserving and, using Tarski's Fixed Point Theorem, we may conclude it has a fixed point. □

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